Parametric Surfaces and their Areas

In this section, we will learn about:

Various types of parametric surfaces

and computing their areas using vector functions.

INTRODUCTION

We describe a space curve by a vector function **r**(*t*) of a single parameter *t.*

Similarly, we can describe a surface by a vector function **r**(*u*, *v*) of two parameters *u* and *v*.

We suppose that

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z\,(u, v)\,\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane.

INTRODUCTION

So x, y, and z—the component functions of **r**—are functions of the two variables u and v with domain D.

PARAMETRIC SURFACE

Equations 2

The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called a parametric surface S.

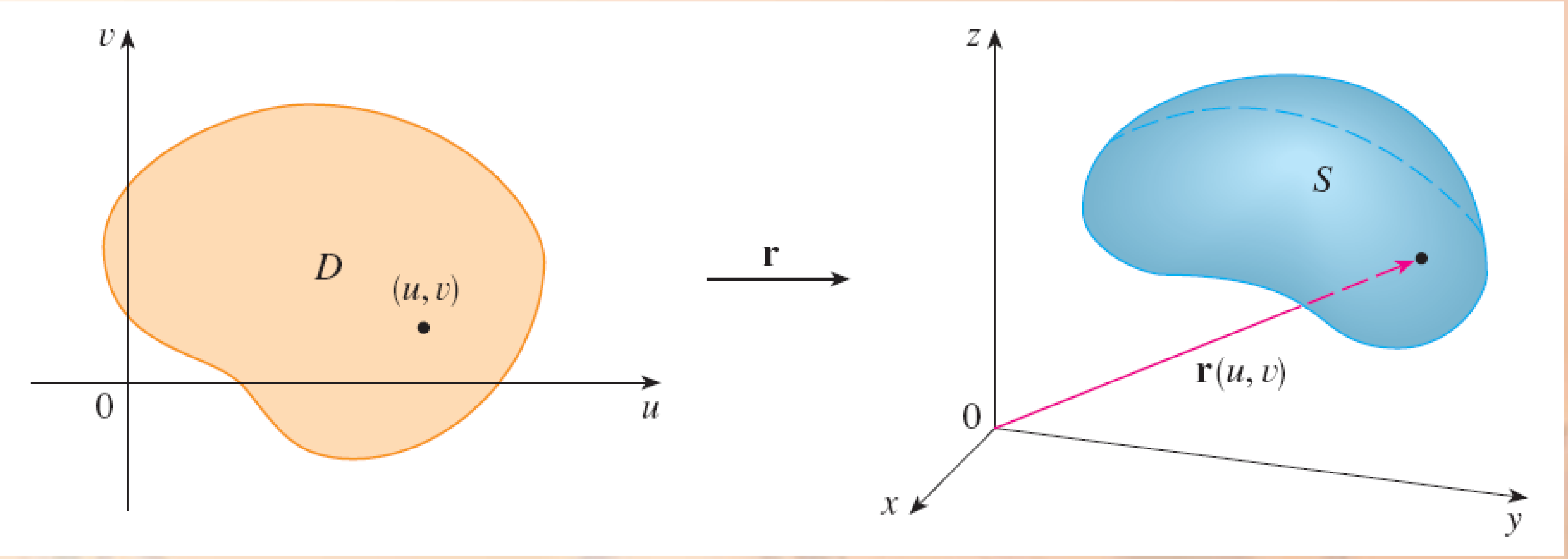
- Equations 2 are called parametric equations of S.

Each choice of u and v gives a point on S.

By making all choices, we get all of S.

PARAMETRIC SURFACES

In other words, the surface *S* is traced out by the tip of the position vector **r**(*u*, *v*) as (*u*, *v*) moves throughout the region *D*.



Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u \, \mathbf{i} + v \, \mathbf{j} + 2 \sin u \, \mathbf{k}$$

- The parametric equations for this surface are:

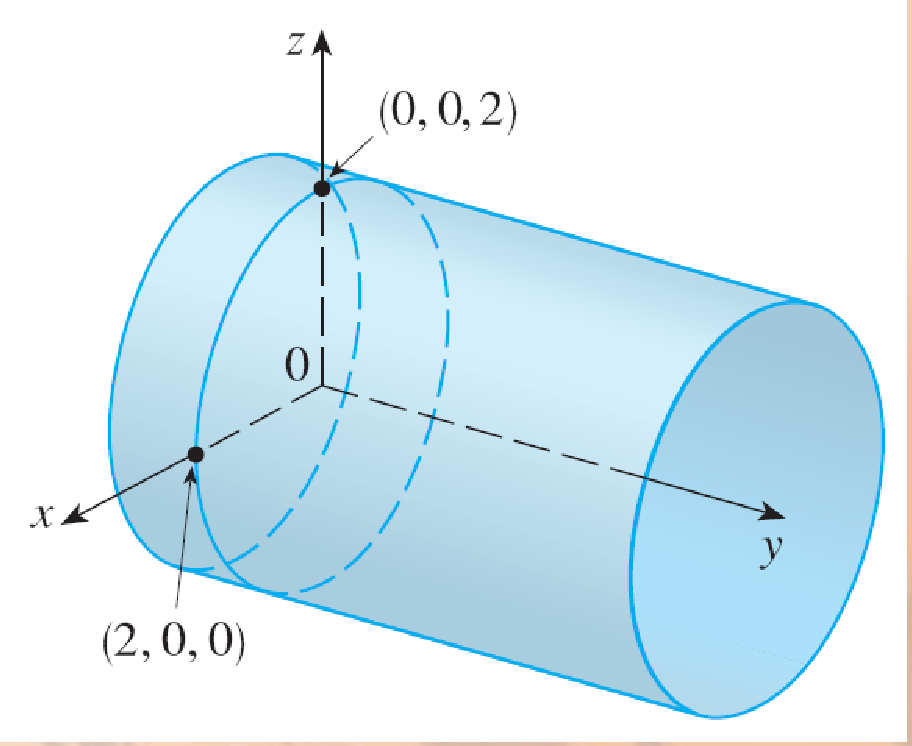
$$x = 2 \cos u \qquad y = v \qquad z = 2 \sin u$$

So, for any point (x, y, z) on the surface, we have:

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u$$
$$= 4$$

- This means that vertical cross-sections parallel to the *xz*-plane (that is, with *y* constant) are all circles with radius 2.

Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y-axis.



In Example 1, we placed no restrictions on the parameters *u* and *v.*

So, we obtained the entire cylinder.

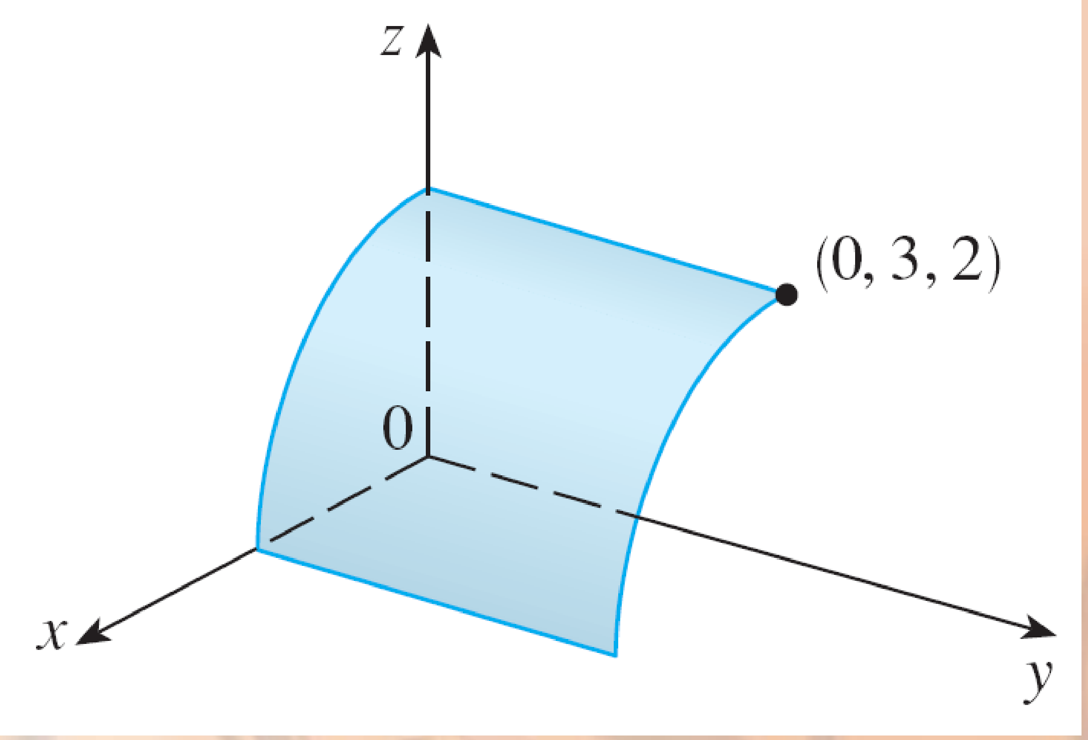
If, for instance, we restrict *u* and *v* by writing the parameter domain as

$$0 \le u \le \pi/2 \qquad 0 \le v \le 3$$

then

$$x \ge 0 \qquad z \ge 0 \qquad 0 \le y \le 3$$

In that case, we get the quarter-cylinder with length 3.

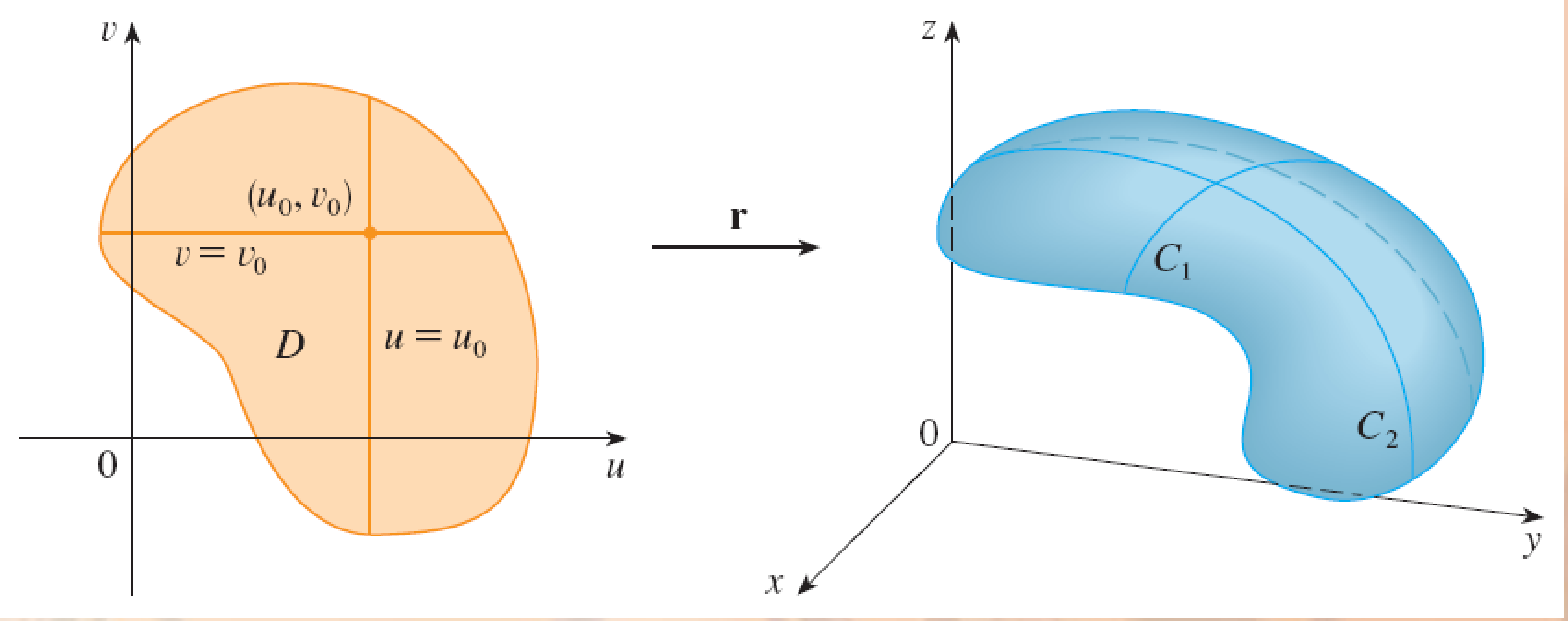


PARAMETRIC SURFACES

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S—one with u constant and the other with v constant.

- These correspond to vertical and horizontal lines in the *uv*-plane.

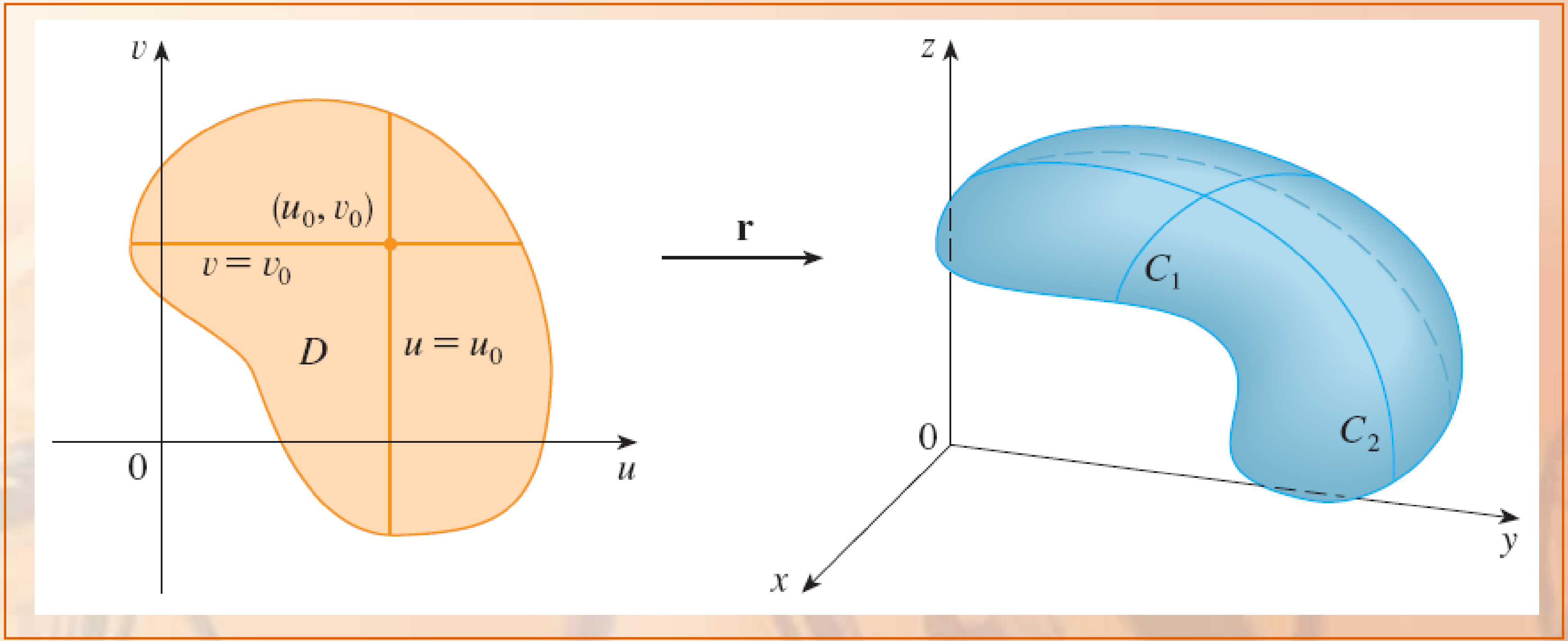
Keeping u constant by putting $u = u_0$, $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S.



GRID CURVES

Similarly, keeping *v* constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on *S*.

- We call these curves grid curves.



GRID CURVES

In Example 1, for instance, the grid curves obtained by:

- Letting *u* be constant are horizontal lines.
- Letting *v* be constant are circles.

GRID CURVES

In fact, when a computer graphs a parametric surface, it usually depicts the surface by plotting these grid curves—as we see in the following example.

Example 2

Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

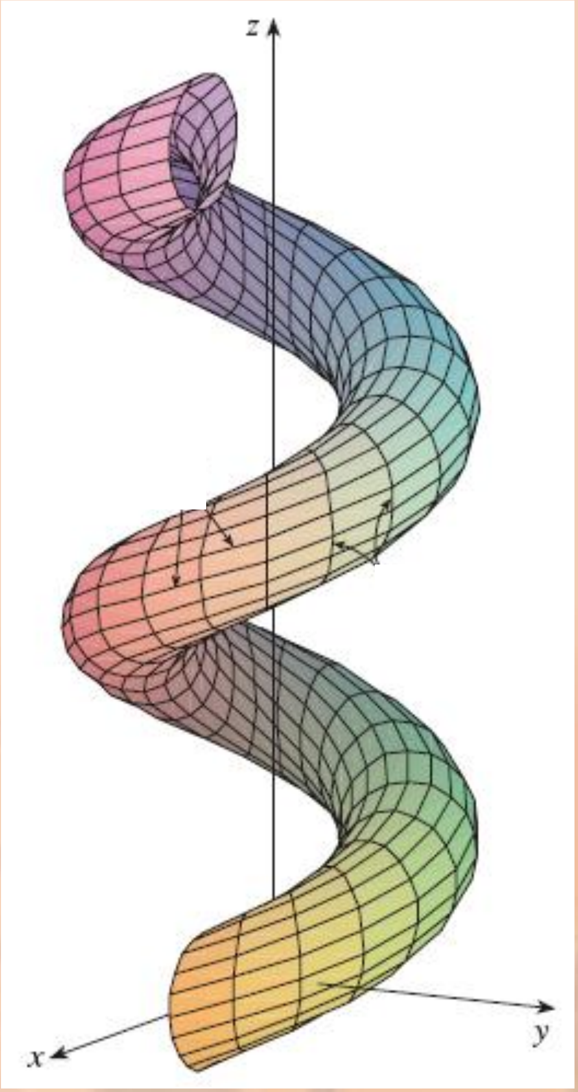
- Which grid curves have u constant?
- Which have v constant?

Example 2

We graph the portion of the surface with parameter domain

$$0 \le u \le 4\pi,\ 0 \le v \le 2\pi$$

- It has the appearance of a spiral tube.



To identify the grid curves, we write the corresponding parametric equations:

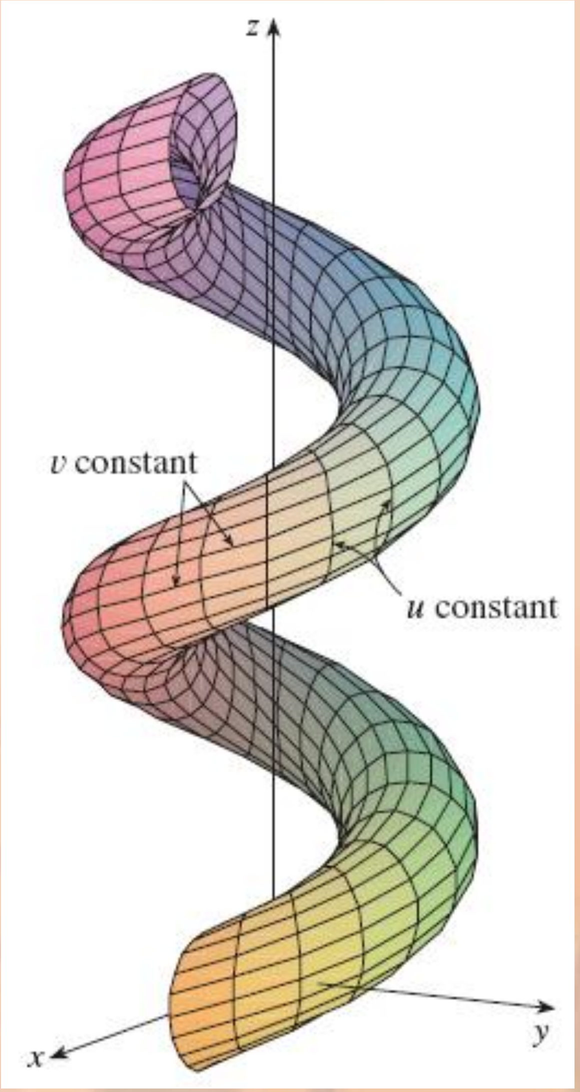
$$x = (2 + \sin v) \cos u$$
$$y = (2 + \sin v) \sin u$$
$$z = u + \cos v$$

If v is constant, then $\sin v$ and $\cos v$ are constant.

- So, the parametric equations resemble those of the helix in Example 4 in Section 13.1

So, the grid curves with *v* constant are the spiral curves.

- We deduce that the grid curves with *u* constant must be the curves that look like circles.



Further evidence for this assertion is that, if u is kept constant, $u = u_0$, then the equation

$$z = u_0 + \cos v$$

shows that the z-values vary from $u_0 - 1$ to $u_0 + 1$.

PARAMETRIC REPRESENTATION

In Examples 1 and 2 we were given a vector equation and asked to graph the corresponding parametric surface.

- In the following examples, however, we are given the more challenging problem of finding a vector function to represent a given surface.
- In the rest of the chapter, we will often need to do exactly that.

Find a vector function that represents the plane that:

- Passes through the point P_0 with position vector $\mathbf{r}_0$.
- Contains two nonparallel vectors **a** and **b**.

PARAMETRIC REPRESENTATIONS **Example 3**

If P is any point in the plane, we can get from P_0 to P by moving a certain distance in the direction of **a** and another distance in the direction of **b**.

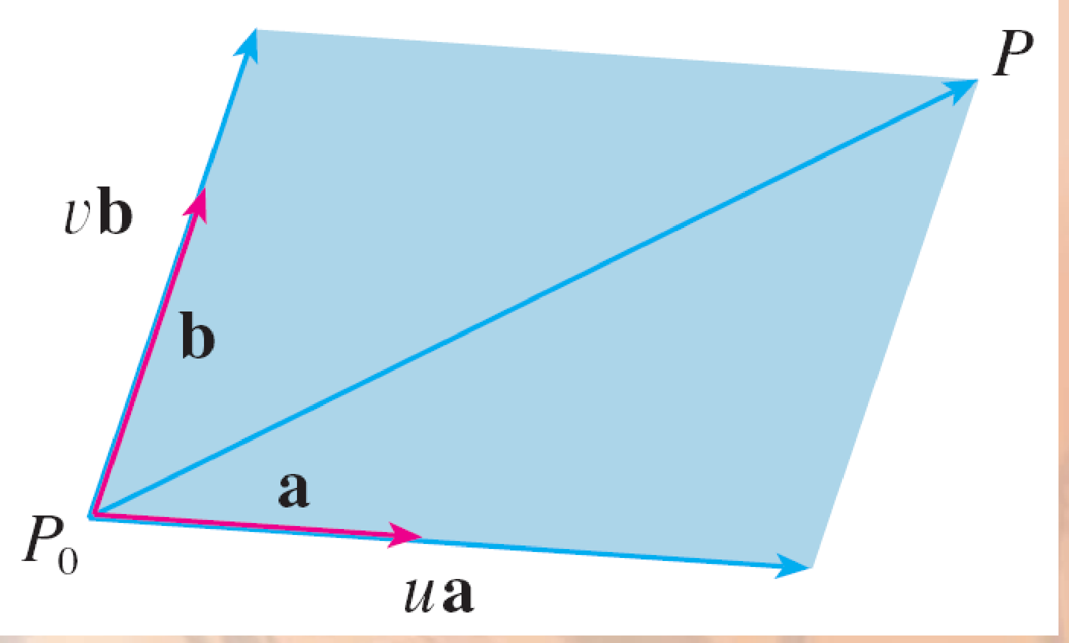
- So, there are scalars u and v such that:

$$\overrightarrow{P_0P} = u\mathbf{a} + v\mathbf{b}$$

PARAMETRIC REPRESENTATIONS **Example 3**

The figure illustrates how this works, by means of the Parallelogram Law, for the case where *u* and *v* are positive.

- See also Exercise 40 in Section 12.2



PARAMETRIC REPRESENTATIONS **Example 3**

If **r** is the position vector of *P*,

then

$$\mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

- So, the vector equation of the plane can be written as:

  $$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

  where *u* and *v* are real numbers.

PARAMETRIC REPRESENTATIONS Example 3

If we write

$$\mathbf{r} = \langle x, y, z \rangle \qquad \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \qquad \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

we can write the parametric equations of the plane through the point (x_0, y_0, z_0) as:

$$x = x_0 + ua_1 + vb_1$$

$$y = y_0 + ua_2 + vb_2$$

$$z = z_0 + ua_3 + vb_3$$

PARAMETRIC REPRESENTATIONS **Example 4**

Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

- The sphere has a simple representation $\rho = a$ in spherical coordinates.
- So, let's choose the angles Φ and θ in spherical coordinates as the parameters (Section 11.8).

PARAMETRIC REPRESENTATIONS **Example 4**

Then, putting $\rho = a$ in the equations for conversion from spherical to rectangular coordinates (Equations 1 in Section 11.8), we obtain:

$$x = a \sin \Phi \cos \theta \qquad y = a \sin \Phi \sin \theta$$

$$z = a \cos \Phi$$

as the parametric equations of the sphere.

PARAMETRIC REPRESENTATIONS **Example 4**

The corresponding vector equation is:

$\mathbf{r}(\Phi, \theta)$

$= a \sin \Phi \cos \theta \, \mathbf{i} + a \sin \Phi \sin \theta \, \mathbf{j} + a \cos \Phi \, \mathbf{k}$

- We have $0 \leq \Phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

- So, the parameter domain is the rectangle

$$D = [0, \pi] \times [0, 2\pi]$$

The grid curves with:

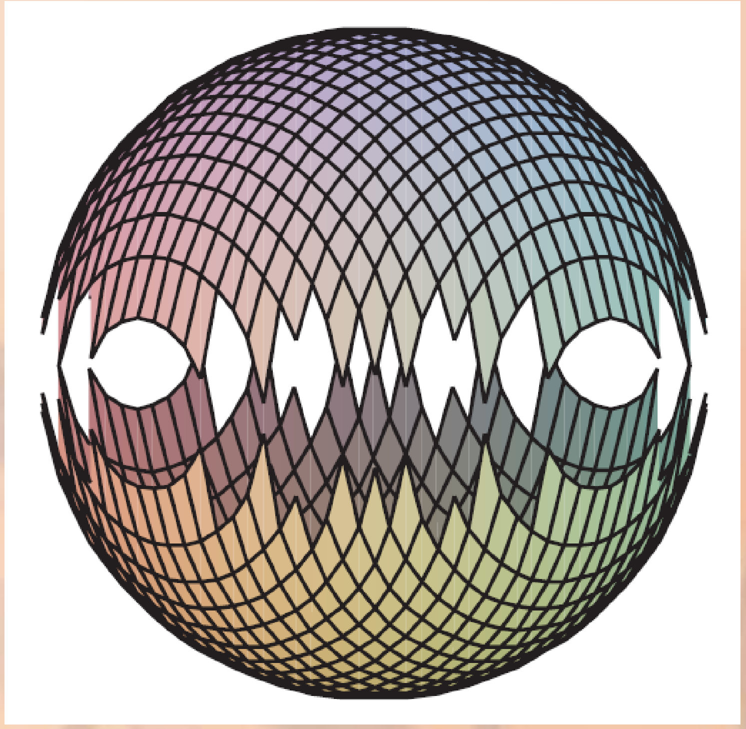
- Φ constant are the circles of constant latitude (including the equator).
- θ constant are the meridians (semicircles), which connect the north and south poles.

APPLICATIONS—COMPUTER GRAPHICS

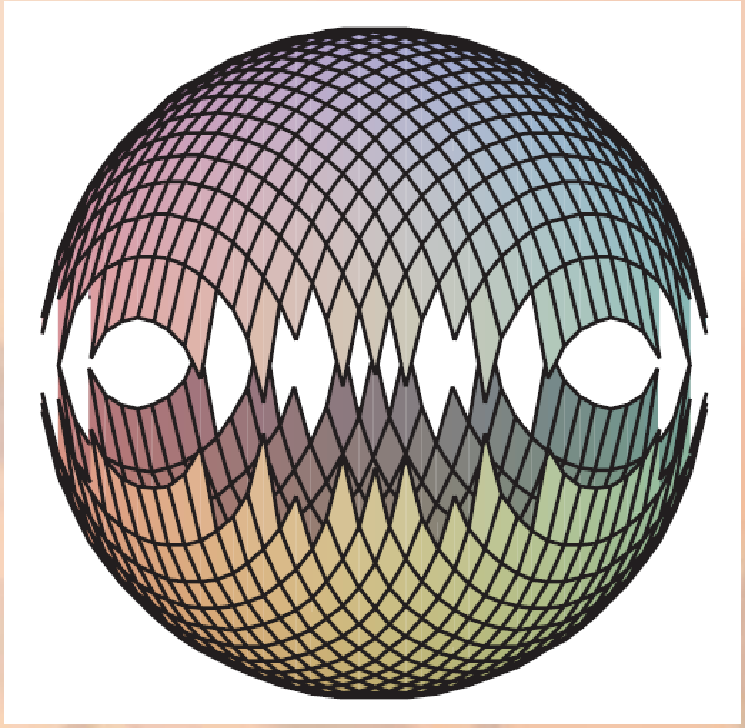
One of the uses of parametric surfaces is in computer graphics.

The figure shows the result of trying to graph the sphere $x^2 + y^2 + z^2 = 1$ by:

- Solving the equation for z.
- Graphing the top and bottom hemispheres separately.

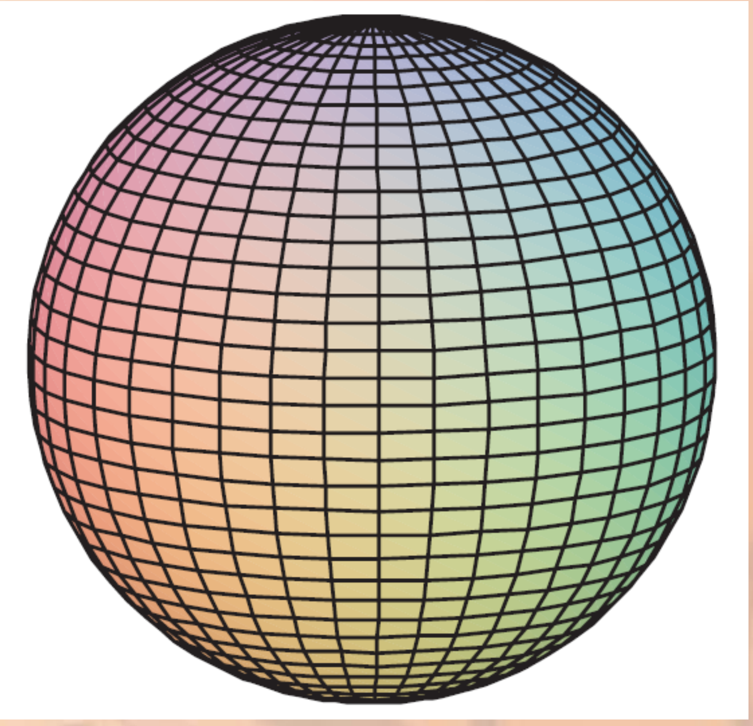


Part of the sphere appears to be missing because of the rectangular grid system used by the computer.



COMPUTER GRAPHICS

The much better picture here was produced by a computer using the parametric equations found in Example 4.



Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad\qquad 0 \le z \le 1$$

- The cylinder has a simple representation $r = 2$ in cylindrical coordinates.

- So, we choose as parameters θ and z in cylindrical coordinates.

PARAMETRIC REPRESENTATIONS **Example 5**

Then the parametric equations of the cylinder are

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = z$$

where:

- $0 \le \theta \le 2\pi$
- $0 \le z \le 1$

Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$

- If we regard x and y as parameters, then the parametric equations are simply

$$x = x \qquad y = y \qquad z = x^2 + 2y^2$$

and the vector equation is

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + 2y^2)\,\mathbf{k}$$

PARAMETRIC REPRESENTATIONS

In general, a surface given as the graph of a function of x and y—an equation of the form $z = f(x, y)$—can always be regarded as a parametric surface by:

- Taking x and y as parameters.
- Writing the parametric equations as

$$x = x \qquad y = y \qquad z = f(x, y)$$

Parametric representations (also called parametrizations) of surfaces are not unique.

- The next example shows two ways to parametrize a cone.

Find a parametric representation for the surface

$$z = 2\sqrt{x^2 + y^2}$$

that is,

the top half of the cone

$$z^2 = 4x^2 + 4y^2$$

One possible representation is obtained by choosing x and y as parameters:

$$x = x \qquad y = y \qquad z = 2\sqrt{x^2 + y^2}$$

- So, the vector equation is:

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + 2\sqrt{x^2 + y^2}\,\mathbf{k}$$

Another representation results from choosing as parameters the polar coordinates r and θ.

- A point (x, y, z) on the cone satisfies:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = 2\sqrt{x^2 + y^2} = 2r$$

So, a vector equation for the cone is

$$\mathbf{r}(r, \theta) = r \cos \theta \, \mathbf{i} + r \sin \theta \, \mathbf{j} + 2r \, \mathbf{k}$$

where:

- $r \geq 0$
- $0 \leq \theta \leq 2\pi$

PARAMETRIZATIONS

For some purposes, the parametric representations in Solutions 1 and 2 are equally good.

In certain situations, though, Solution 2 might be preferable.

For instance, if we are interested only in the part of the cone that lies below the plane $z = 1$, all we have to do in Solution 2 is change the parameter domain to:

$$0 \leq r \leq \tfrac{1}{2} \qquad 0 \leq \theta \leq 2\pi$$

SURFACES OF REVOLUTION

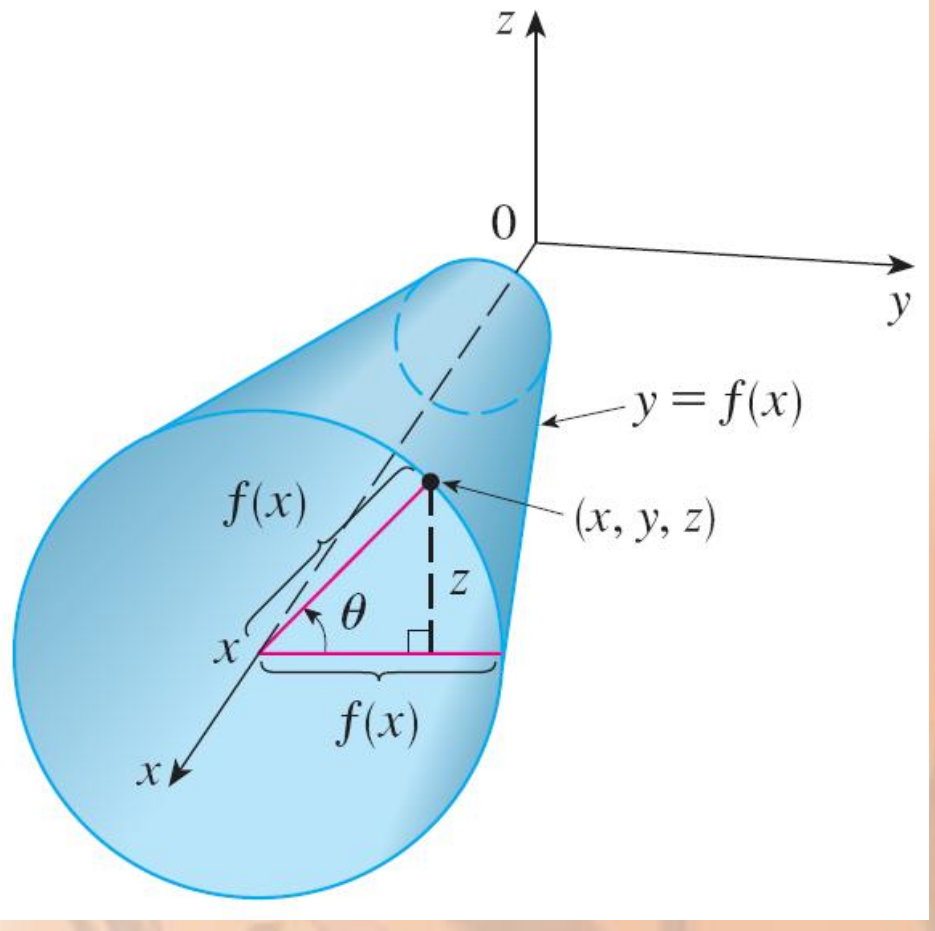
Surfaces of revolution can be represented parametrically and thus graphed using a computer.

For instance, let's consider the surface S obtained by rotating the curve

$$y = f(x) \qquad a \leq x \leq b$$

about the x-axis, where $f(x) \geq 0$.

Let θ be the angle of rotation as shown.



SURFACES OF REVOLUTION

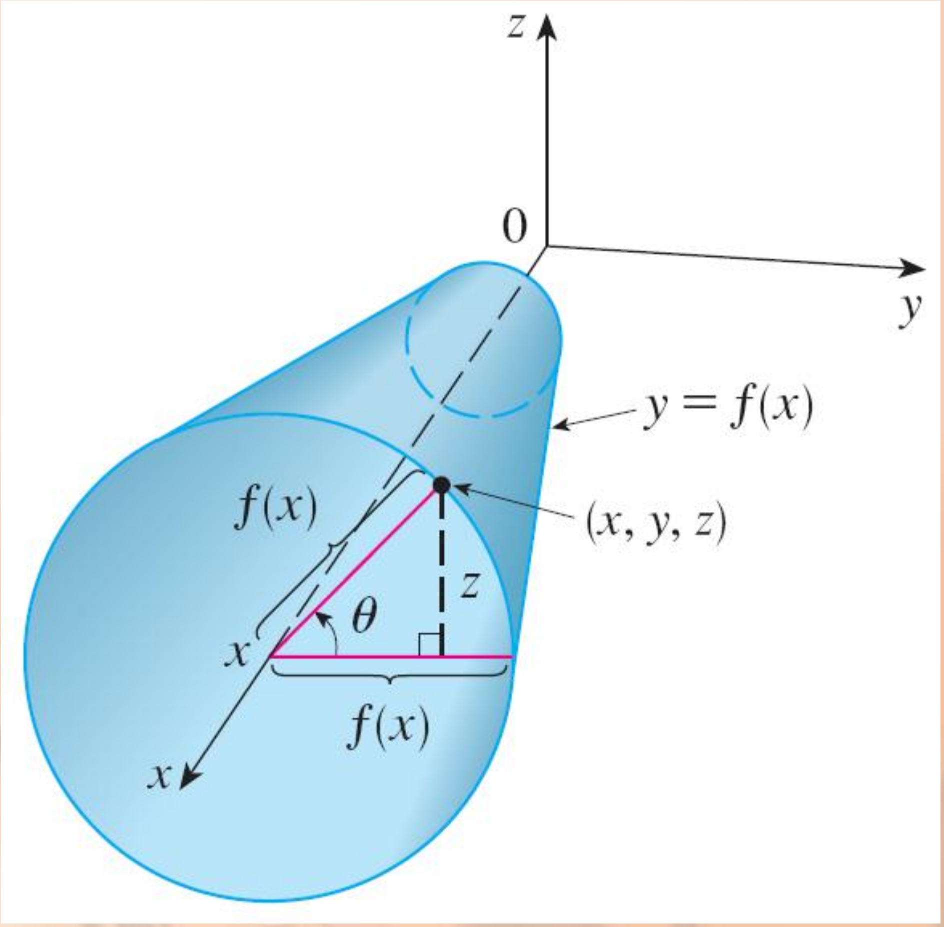
Equations 3

If (x, y, z) is a point on S, then

$$x = x$$

$$y = f(x)\cos\theta$$

$$z = f(x)\sin\theta$$



Thus, we take x and θ as parameters and regard Equations 3 as parametric equations of S.

- The parameter domain is given by:

$$a \leq x \leq b \qquad\qquad 0 \leq \theta \leq 2\pi$$

Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \le x \le 2\pi$, about the x-axis.

Use these equations to graph the surface of revolution.

From Equations 3,

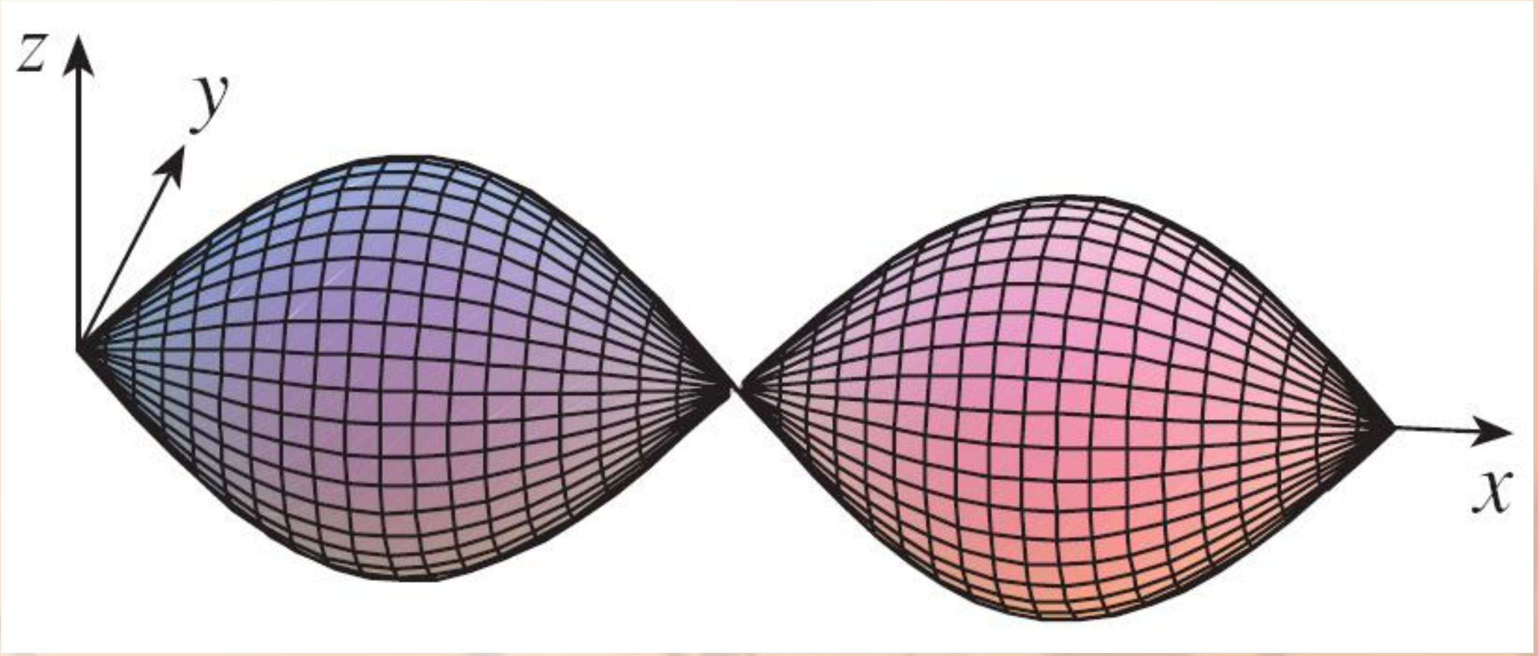
- The parametric equations are:

$$x = x \qquad y = \sin x \cos \theta \qquad z = \sin x \sin \theta$$

- The parameter domain is:

$$0 \le x \le 2\pi \qquad 0 \le \theta \le 2\pi$$

Using a computer to plot these equations and rotate the image, we obtain this graph.



We can adapt Equations 3 to represent a surface obtained through revolution about the *y*- or *z*-axis.

- See Exercise 30.