

The background features a close-up, slightly blurred image of a clock's pendulum mechanism. The pendulum consists of a metal rod with two circular weights, one above and one below the pivot. The clock face is visible in the background, showing Roman numerals. On the right side of the image, the number '16' is displayed in a large, light orange, sans-serif font.

16

A solid orange rectangular box with a thin black border is positioned horizontally across the middle of the image. Inside the box, the words 'VECTOR CALCULUS' are written in a white, bold, sans-serif font.

VECTOR CALCULUS

16.3

Fundamental Theorem for Line Integrals

In this section, we will learn about:

The Fundamental Theorem for line integrals
and determining conservative vector fields.

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$.

Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C .

Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

CONSERVATIVE VECTOR FIELDS Theorem 5

If

$$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D , then, throughout D ,

we have:
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

CONSERVATIVE VECTOR FIELDS Theorem 6

Let $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ be a vector field on an open simply-connected region D .

Suppose that P and Q have continuous first-order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D .

- Then, F is conservative.

CONSERVATIVE VECTOR FIELDS Example 2

Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y) \mathbf{i} + (x - 2) \mathbf{j}$$

is conservative.

- Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$.
- Then, $\frac{\partial P}{\partial y} = -1$ $\frac{\partial Q}{\partial x} = 1$
- As $\partial P/\partial y \neq \partial Q/\partial x$, \mathbf{F} is not conservative by Theorem 5.

FINDING POTENTIAL FUNCTION Example 4

a. If $\mathbf{F}(x, y) = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$,
find a function f such that $\mathbf{F} = \nabla f$.

b. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$,
where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} \quad 0 \leq t \leq \pi$$

FINDING POTENTIAL FUNCTION E. g. 4 a—Eqns. 7 & 8

From Example 3, we know that \mathbf{F} is conservative.

So, there exists a function f with $\nabla f = \mathbf{F}$, that is,

$$f_x(x, y) = 3 + 2xy$$

$$f_y(x, y) = x^2 - 3y^2$$

FINDING POTENTIAL FUNCTION E. g. 4 a—Eqn. 9

Integrating Equation 7 with respect to x ,
we obtain:

$$f(x, y) = 3x + x^2y + g(y)$$

- Notice that the constant of integration is a constant with respect to x , that is, a function of y , which we have called $g(y)$.

FINDING POTENTIAL FUNCTION E. g 4 a—Eqn. 10

Next, we differentiate both sides of Equation 9 with respect to y :

$$f_y(x, y) = x^2 + g'(y)$$

FINDING POTENTIAL FUNCTION

Example 4 a

Comparing Equations 8 and 10,
we see that:

$$g'(y) = -3y^2$$

- Integrating with respect to y ,
we have:

$$g(y) = -y^3 + K$$

where K is a constant.

FINDING POTENTIAL FUNCTION

Example 4 a

Putting this in Equation 9,
we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

FINDING POTENTIAL FUNCTION Example 4 b

To use Theorem 2, all we have to know are the initial and terminal points of C , namely,

$$\mathbf{r}(0) = (0, 1)$$

$$\mathbf{r}(\pi) = (0, -e^\pi)$$

FINDING POTENTIAL FUNCTION Example 4 b

In the expression for $f(x, y)$ in part a, any value of the constant K will do.

- So, let's choose $K = 0$.

Then, we have:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^\pi) - f(0, 1) \\ &= e^{3\pi} - (-1) = e^{3\pi} + 1\end{aligned}$$

- This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 12.2