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VECTOR CALCULUS

16.3

Fundamental Theorem for Line Integrals

In this section, we will learn about:

The Fundamental Theorem for line integrals and determining conservative vector fields.

FT FOR LINE INTEGRALS

Theorem 2

Let *C* be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$.

Let *f* be a differentiable function of two or three variables whose gradient vector ∇f is continuous on *C*.

Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f\left(\mathbf{r}\left(b\right)\right) - f\left(\mathbf{r}\left(a\right)\right)$$

CONSERVATIVE VECTOR FIELDS **Theorem 5**

If

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$$

is a conservative vector field, where *P* and *Q* have continuous first-order partial derivatives on a domain *D*, then, throughout *D*,

we have:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

CONSERVATIVE VECTOR FIELDS **Theorem 6**

Let **F** = *P* **i** + *Q* **j** be a vector field on an open simply-connected region *D*.

Suppose that *P* and *Q* have continuous first-order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout *D.*

- Then, *F* is conservative.

Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

- Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$.
- Then, $\dfrac{\partial P}{\partial y} = -1 \qquad \dfrac{\partial Q}{\partial x} = 1$
- As $\partial P/\partial y \neq \partial Q/\partial x$, **F** is not conservative by Theorem 5.

a. If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

b. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \le t \le \pi$

From Example 3, we know that **F** is conservative.

So, there exists a function *f* with $\nabla f = \mathbf{F}$, that is,

$$f_x(x, y) = 3 + 2xy$$

$$f_y(x, y) = x^2 - 3y^2$$

Integrating Equation 7 with respect to *x*, we obtain:

$$f(x, y) = 3x + x^2y + g(y)$$

- Notice that the constant of integration is a constant with respect to *x*, that is, a function of *y*, which we have called $g(y)$.

Next, we differentiate both sides of Equation 9 with respect to *y*:

$$f_y(x, y) = x^2 + g'(y)$$

Comparing Equations 8 and 10,

we see that:

$$g'(y) = -3y^2$$

- Integrating with respect to y, we have:

$$g(y) = -y^3 + K$$

where K is a constant.

FINDING POTENTIAL FUNCTION **Example 4 a**

Putting this in Equation 9,

we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

To use Theorem 2, all we have to know are the initial and terminal points of C, namely,

$$\mathbf{r}(0) = (0, 1)$$

$$\mathbf{r}(\pi) = (0, -e^{\pi})$$

In the expression for $f(x, y)$ in part a, any value of the constant K will do.

- So, let's choose $K = 0$.

FINDING POTENTIAL FUNCTION Example 4 b

Then, we have:

$$\int_C \mathbf{F} \cdot dr = \int_C \nabla f \cdot dr = f\left(0, -e^{\pi}\right) - f\left(0,1\right)$$

$$= e^{3\pi} - \left(-1\right) = e^{3\pi} + 1$$

- This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 12.2