CHAPTER 12 MULTIPLE INTEGRALS

MULTIPLE INTEGRALS

MULTIPLE INTEGRALS

- Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral.
  - However, the Fundamental Theorem of Calculus (FTC) provides a much easier method.

INTRODUCTION

- The evaluation of double integrals from first principles is even more difficult.
- Once we have expressed a double integral as an iterated integral, we can then evaluate it by calculating two single integrals.

INTRODUCTION

- Suppose that f is a function of two variables that is integrable on the rectangle

$$R = [a, b] \times [c, d]$$

- We use the notation $\int_c^d f(x, y)\, dy$ to mean:
  - x is held fixed.
  - $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$.

PARTIAL INTEGRATION

- This procedure is called *partial integration with respect to y*.
  - Notice its similarity to partial differentiation.

- Now, $\int_c^d f(x, y)\,dy$ is a number that depends on the value of *x*.
- So, it defines a function of *x*:

$$A(x) = \int_c^d f(x, y)\,dy$$

PARTIAL INTEGRATION

- If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

ITERATED INTEGRAL

- The integral on the right side of Equation 7 is called an **iterated integral**.
  - Usually, the brackets are omitted.

ITERATED INTEGRALS

- Thus,

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

means that:

- First, we integrate with respect to y from c to d.
- Then, we integrate with respect to x from a to b.

ITERATED INTEGRALS

- Similarly, the iterated integral

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

means that:

  - First, we integrate with respect to x (holding y fixed) from $x = a$ to $x = b$.
  - Then, we integrate the resulting function of y with respect to y from $y = c$ to $y = d$.

- Notice that, in both Equations 8 and 9, we work from the inside out.

Example 4

- Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Example 4(a) SOLUTION

❖ Regarding *x* as a constant, we obtain:

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2}$$

$$= x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right)$$

$$= \tfrac{3}{2} x^2$$

Example 4(a) SOLUTION

- Thus, the function *A* in the preceding discussion is given by

$$A(x) = \tfrac{3}{2}x^2$$

in this example.

Example 4(a) SOLUTION

❖We now integrate this function of *x* from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2 dx = \frac{x^3}{2} \Big]_0^3$$

$$= \frac{27}{2}$$

Example 4(b) SOLUTION

- Here, we first integrate with respect to *x*:

$$\int_1^2 \int_0^3 x^2 y\, dx\, dy = \int_1^2 \left[\int_0^3 x^2 y\, dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y\, dy = 9\frac{y^2}{2}\Big]_1^2 = \frac{27}{2}$$

ITERATED INTEGRALS

- Notice that, in Example 4, we obtained the same answer whether we integrated with respect to y or x first.
- In general, it turns out (see Theorem 10) that the two iterated integrals in Equations 8 and 9 are always equal.
  - That is, the order of integration does not matter.
  - This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.

ITERATED INTEGRALS

- The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

FUBUNI'S THEOREM

If f is continuous on the rectangle

$$R = \{(x, y) \mid a \le x \le b,\ c \le y \le d$$

then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

FUBUNI'S THEOREM

- Theorem 10 is named after the Italian mathematician Guido Fubini (1879–1943), who proved a very general version of this theorem in 1907.
  - However, the version for continuous functions was known to the French mathematician Augustin-Louis Cauchy almost a century earlier.

FUBUNI'S THEOREM

- The proof of Fubini's Theorem is too difficult to include in this book.
- However, we can at least give an intuitive indication of why it is true for the case where $f(x, y) \geq 0$.

FUBUNI'S THEOREM

- Recall that, if f is positive, then we can interpret the double integral

$$\iint_R f(x, y)\, dA$$

as:

  - The volume V of the solid S that lies above R and under the surface $z = f(x, y)$.

FUBUNI'S THEOREM

- However, we have another formula that we used for volume in Chapter 7, namely,

$$V = \int_a^b A(x)\,dx$$

where:

  - $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis.

FUBUNI'S THEOREM

- From Figure 11, you can see that $A(x)$ is the area under the curve C whose equation is

$$z = f(x, y)$$

where:

  - x is held constant
  - $c \leq y \leq d$

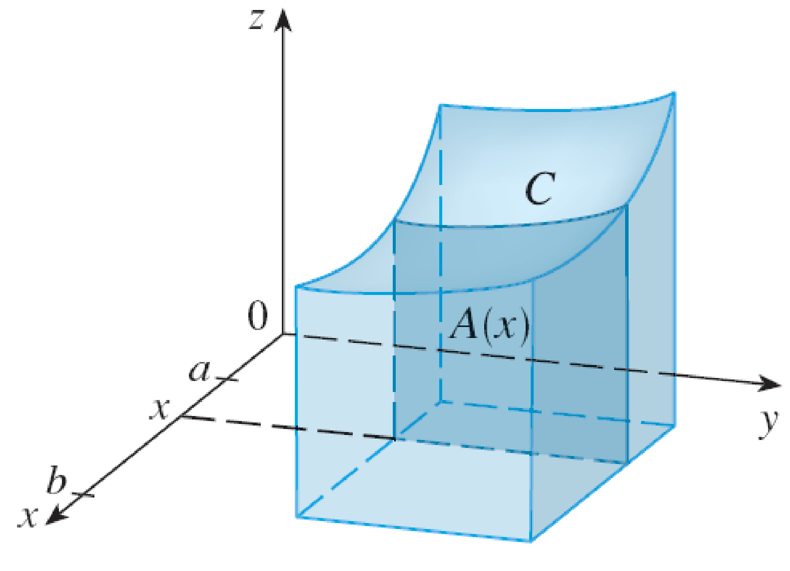

FIGURE 11

FUBUNI'S THEOREM

- Therefore,

$$A(x) = \int_c^d f(x, y)\, dy$$

Then, we have:

$$\iint_R f(x, y)\, dA = V = \int_a^b A(x)\, dx$$

$$= \int_a^b \int_c^d f(x, y)\, dy\, dx$$

FUBUNI'S THEOREM

- A similar argument, using cross-sections perpendicular to the *y*-axis as in Figure 12, shows that:

$$\iint_R f(x,y)\,dA$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

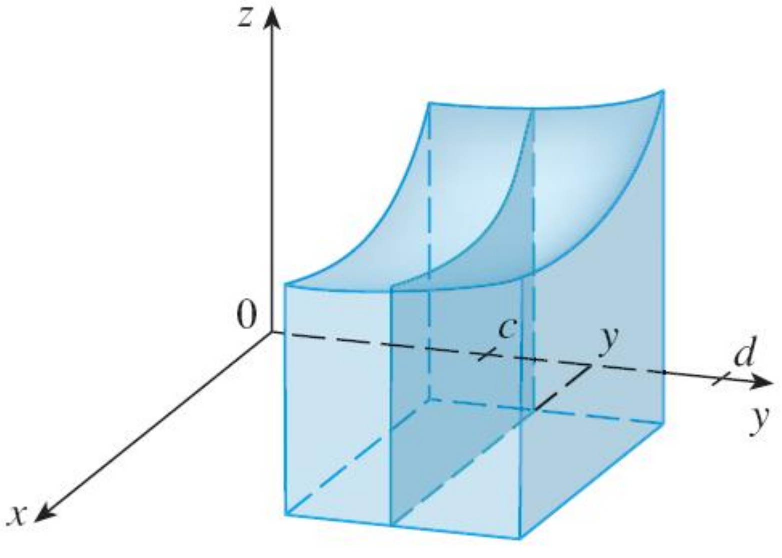

FIGURE 12

Example 5

- Evaluate the double integral

$$\iint_R (x - 3y^2)\, dA$$

where

$$R = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$$

  - Compare with Example 3.

Example 5 SOLUTION 1

- Fubini's Theorem gives:

$$\iint_R (x-3y^2)\,dA = \int_0^2 \int_1^2 (x-3y^2)\,dy\,dx$$

$$= \int_0^2 \left[xy - y^3 \right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x-7)\,dx = \frac{x^2}{2} - 7x \Big]_0^2$$

$$= -12$$

Example 5 SOLUTION 2

- This time, we first integrate with respect to x:

$$\iint_R (x-3y^2)\,dA = \int_1^2 \int_0^2 (x-3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2 \right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2-6y^2)\,dy = 2y - 2y^3 \Big]_1^2$$

$$= -12$$

FUBUNI'S THEOREM

- Notice the negative answer in Example 2.
- Nothing is wrong with that.
  - The function *f* in the example is not a positive function.
  - So, its integral doesn't represent a volume.

FUBUNI'S THEOREM

- From Figure 13, we see that f is always *negative* on R.
  - Thus, the value of the integral is the negative of the volume that lies above the graph of f and *below* R.

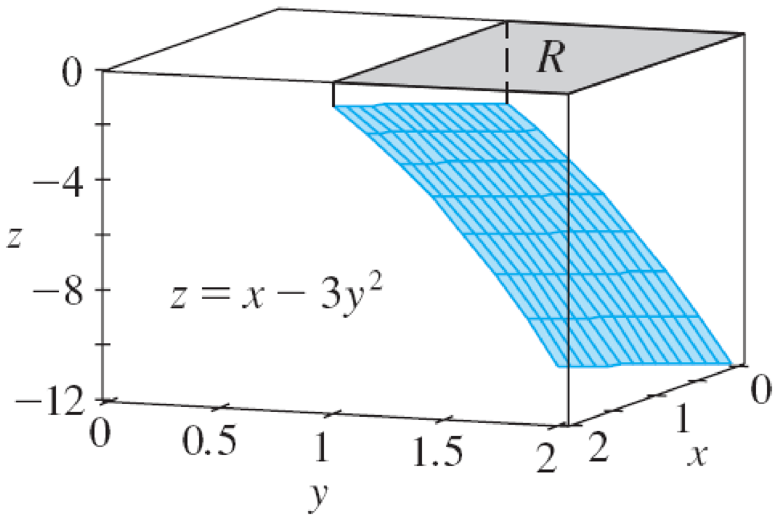

FIGURE 13

Example 6

- Evaluate

$$\iint_R y\sin(xy)\,dA$$

where

$$R = [1, 2] \times [0, \pi]$$

Example 6 SOLUTION

- If we first integrate with respect to *x*, we get:

$$\iint_R y\sin(xy)\,dA = \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$

$$= \int_0^{\pi}[-\cos(xy)]_{x=1}^{x=2}\,dy$$

$$= \int_0^{\pi}(-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^{\pi} = 0$$

NOTE

- If we first integrate with respect to *y* in Example 6, we get

$$\iint_R y\sin(xy)\,dA = \int_1^2 \int_0^{\pi} y\sin(xy)\,dy\,dx$$

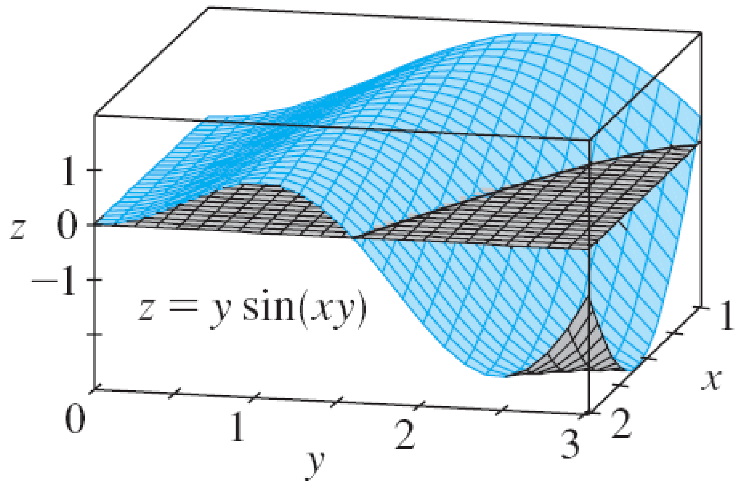

FIGURE 14

NOTE

- However this order of integration is much more difficult than the method given in the example because it involves integration by parts twice.
- Therefore, when we evaluate double integrals it is wise to choose the order of integration that gives simpler integrals.

Example 7

- Find the volume of the solid S that is bounded by:
  - The elliptic paraboloid $x^2 + 2y^2 + z = 16$
  - The planes $x = 2$ and $y = 2$
  - The three coordinate planes

Example 7 SOLUTION

- We first observe that S is the solid that lies:
  - Under the surface $z = 16 - x^2 - 2y^2$
  - Above the square $R = [0, 2] \times [0, 2]$
  - See Figure 15

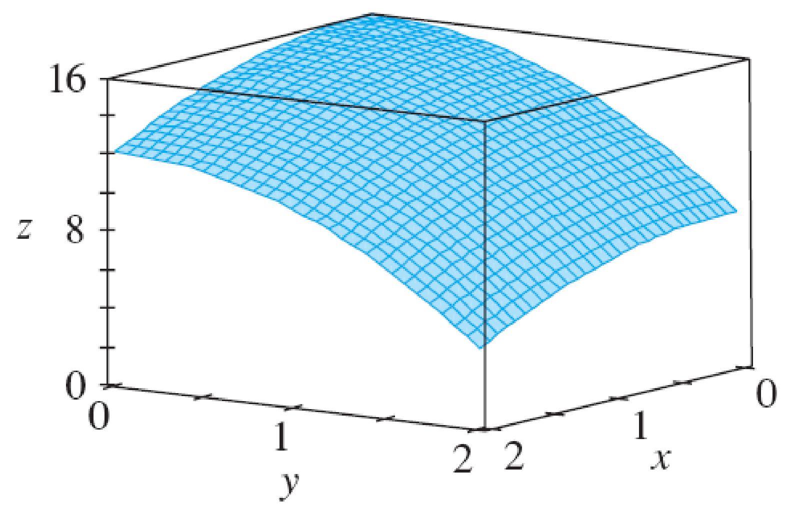

FIGURE 15

Example 7 SOLUTION

- This solid was considered in Example 1.
- Now, however, we are in a position to evaluate the double integral using Fubini's Theorem.

Example 7 SOLUTION

❖ Thus,

$$V = \iint_R (16 - x^2 - 2y^2)\, dA$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \tfrac{1}{3}x^3 - 2y^2 x\right]_{x=0}^{x=2} dy$$

$$= \int_0^2 \left(\tfrac{88}{3} - 4y^2\right) dy$$

$$= \left[\tfrac{88}{3}y - \tfrac{4}{3}y^3\right]_0^2 = 48$$

ITERATED INTEGRALS

- Consider the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only.
  - Then, the double integral of f can be written in a particularly simple form.

ITERATED INTEGRALS

- To be specific, suppose that:
  - $f(x, y) = g(x)h(y)$
  - $R = [a, b] \times [c, d]$
- Then, Fubini's Theorem gives:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy$$

$$= \int_c^d \left[\int_a^b g(x)h(y)\,dx \right] dy$$

ITERATED INTEGRALS

- In the inner integral, *y* is a constant.
- So, *h*(*y*) is a constant and we can write:

$$\int_c^d \left[\int_a^b g(x)h(y)\,dx \right] dy = \int_c^d \left[h(y) \left(\int_a^b g(x)\,dx \right) \right] dy$$

$$= \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant.

ITERATED INTEGRALS

- Hence, in this case, the double integral of *f* can be written as the product of two single integrals:

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

where $R = [a, b] \times [c, d]$.

Example 8

❖ If $R = [0, \pi/2] \times [0, \pi/2]$, then, by Equation 5,

$$\iint_R \sin x \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= \left[-\cos x\right]_0^{\pi/2} \left[\sin y\right]_0^{\pi/2}$$

$$= 1 \cdot 1 = 1$$

ITERATED INTEGRALS

- The function $f(x, y) = \sin x \cos y$ in Example 8 is positive on R.
- So, the integral represents the volume of the solid that lies above R and below the graph of f shown in Figure 16.

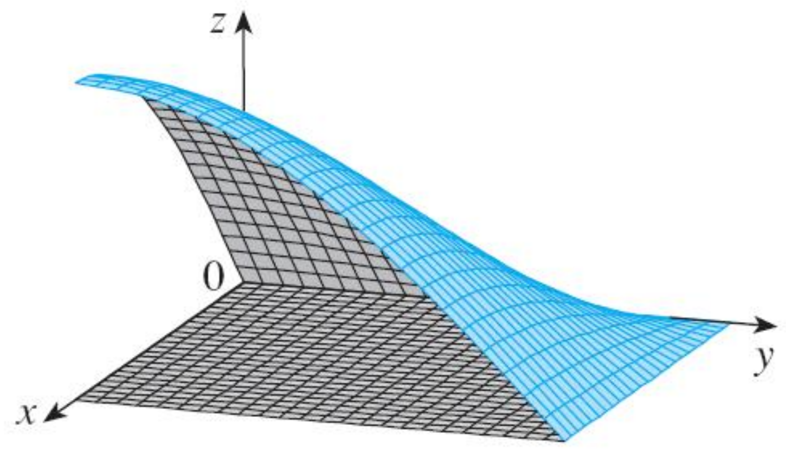

FIGURE 16

PROPERTIES OF DOUBLE INTEGRALS

- We list here three properties of double integrals that can be proved in the same man-ner as in Section 4.2.

PROPERTIES OF DOUBLE INTEGRALS

- We assume that all of the integrals exist. Properties 12 and 13are referred to as the *linearity* of the integral.

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA \quad \text{where } c \text{ is a constant}$$

PROPERTIES OF DOUBLE INTEGRALS

- If $f(x, y) \geq g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$