CHAPTER 3 APPLICATIONS OF DIFFERENTIATION

SECTION 3.5

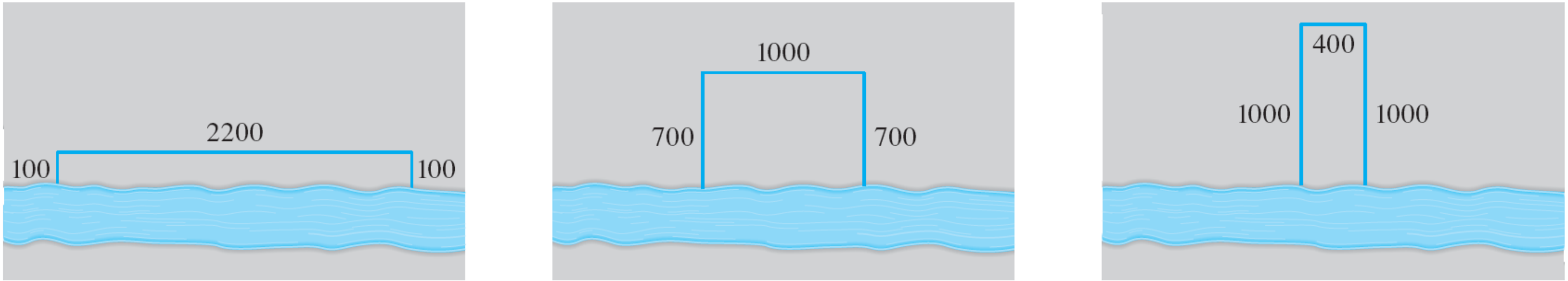
OPTIMIZATION PROBLEMS

Example 1

- A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river.
  - What are the dimensions of the field that has the largest area?
- SOLUTION
  - In order to get a feeling for what is happening in the problem, let's experiment with some special cases.

Example 1 SOLUTION

❖Figure 1 shows three possible ways of laying out the 2400 ft of fencing.



Area = 100 · 2200 = 220,000 ft^2

Area = 700 · 1000 = 700,000 ft^2

Area = 1000 · 400 = 400,000 ft^2

FIGURE 1

Example 1 SOLUTION

- We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas.
  - It seems plausible that there is some intermediate configuration that produces the largest area.

Example 1 SOLUTION

- Figure 2 illustrates the general case.
- We wish to maximize the area A of the rectangle.
  - Let x and y be the depth and width of the rectangle (in feet).
  - Then, we express A in terms of x and y: $A = xy$

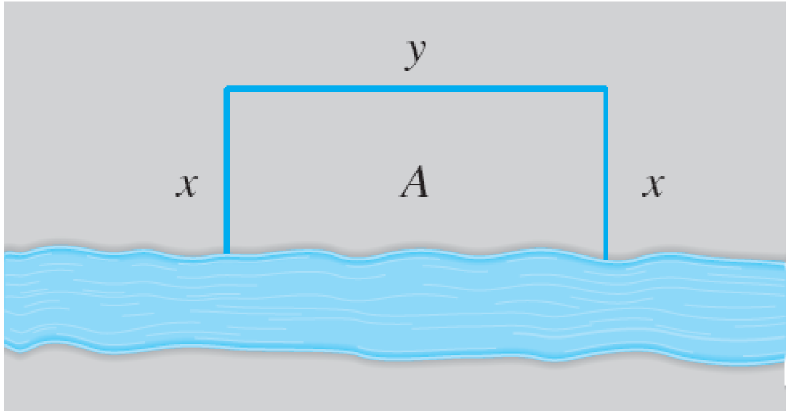


FIGURE 2

Example 1 SOLUTION

- We want to express A as a function of just one variable.
  - So, we eliminate y by expressing it in terms of x.
  - To do this, we use the given information that the total length of the fencing is 2400 ft.
  - Thus, $2x + y = 2400$

Example 1 SOLUTION

- From that equation, we have:

$$y = 2400 - 2x$$

- This gives:

$$A = x(2400 - 2x) = 2400x - 2x^2$$

  - Note that $x \geq 0$ and $x \leq 1200$ (otherwise $A < 0$).

- So, the function that we wish to maximize is:

$$A(x) = 2400x - 2x^2 \qquad 0 \leq x \leq 1200$$

  - The derivative is: $A'(x) = 2400 - 4x$
  - So, to find the critical numbers, we solve: $2400 - 4x = 0$
  - This gives: $x = 600$

Example 1 SOLUTION

- The maximum value of A must occur either at that critical number or at an endpoint of the interval.
  - $A(0) = 0$; $A(600) = 720{,}000$; and $A(1200) = 0$
  - So, the Closed Interval Method gives the maximum value as:

$$A(600) = 720{,}000$$

- Alternatively, we could have observed that

$$A''(x) = -4 < 0 \quad \text{for all } x$$

- So, A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.

Example 1 SOLUTION

- Thus, the rectangular field should be:
  - 600 ft deep
  - 1200 ft wide

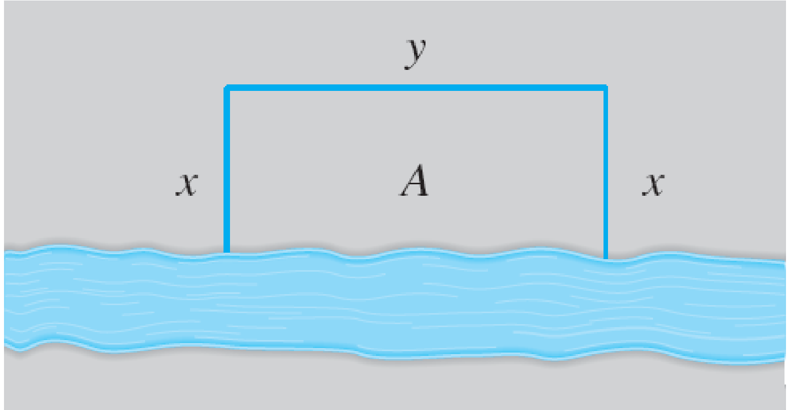


FIGURE 2

Example 2

- A cylindrical can is to be made to hold 1 L of oil.
  - Find the dimensions that will minimize the cost of the metal to manufacture the can.
- SOLUTION
  - Draw the diagram as in Figure 3, where r is the radius and h the height (both in centimeters).

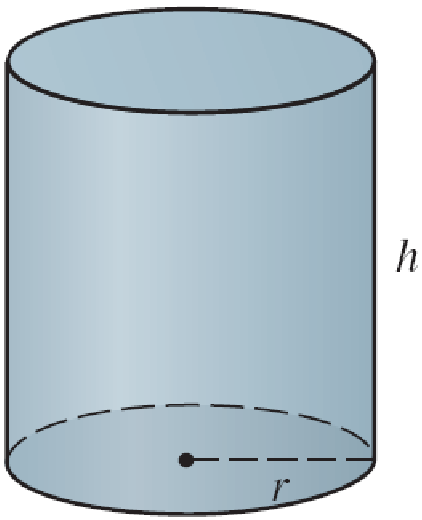


FIGURE 3

Example 2 SOLUTION

- To minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides.)
- From Figure 4 that we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h.
- So, the surface area is:

$$A = 2\pi r^2 + 2\pi rh$$

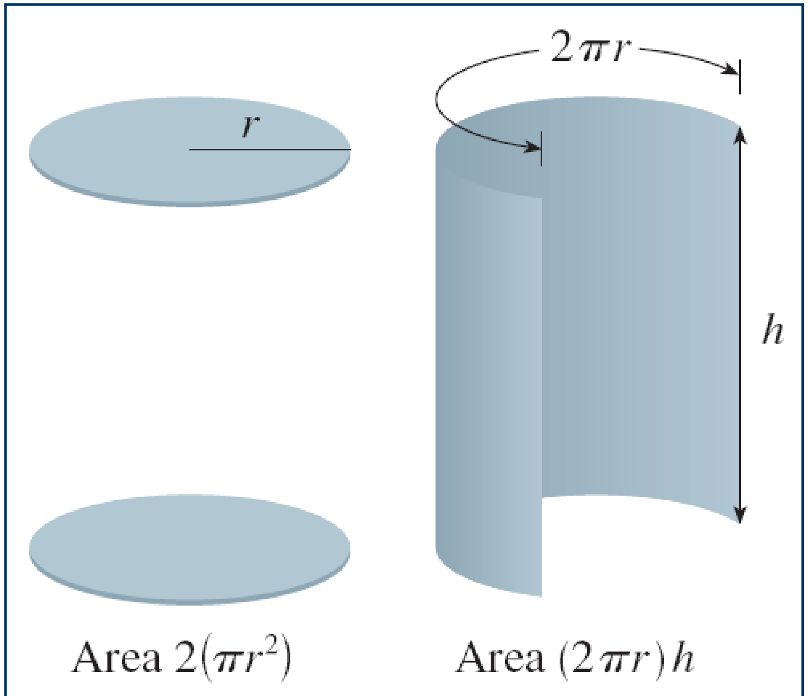


FIGURE 4

Example 2 SOLUTION

- To eliminate *h*, we use the fact that the volume is given as 1 L, which we take to be 1000 cm^3.
  - Thus, $\pi r^2 h = 1000$
  - This gives $h = 1000/(\pi r^2)$
- Substituting this in the expression for *A* gives:

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

- So, the function that we want to minimize is:

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

Example 2 SOLUTION

- To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

- Then, $A'(r) = 0$ when $\pi r^3 = 500$
- So, the only critical number is: $r = \sqrt[3]{500/\pi}$

Example 2 SOLUTION

- As the domain of A is $(0, \infty)$, we can't use the argument of Example 1 concerning endpoints.
  - However, we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$
  - So, A is decreasing for all r to the left of the critical number and increasing for all r to the right.
  - Thus, $r = \sqrt[3]{500/\pi}$ must give rise to an absolute minimum.

Example 2 SOLUTION

- Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$.
  - So, there must be a minimum value of $A(r)$, which must occur at the critical number.
  - See Figure 5.

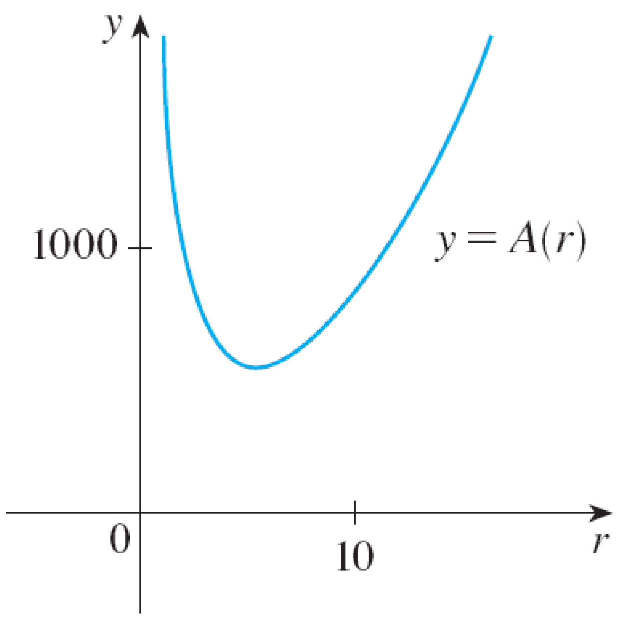


FIGURE 5

Example 2 SOLUTION

- The value of *h* corresponding to is:

$$r = \sqrt[3]{500/\pi}$$

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$