Definition 2

Then, the definite integral of *f* from *a* to *b* is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists.

If it does exist, we say *f* is integrable on [*a*, *b*].

Note 1

The symbol $\int$ was introduced by Leibniz and is called an integral sign.

- It is an elongated *S*.
- It was chosen because an integral is a limit of sums.

The procedure of calculating an integral is called integration.

DEFINITE INTEGRAL $\int_a^b f(x)\,dx$ **Note 2**

The definite integral $\int_a^b f(x)dx$ is a number.

It does not depend on *x*.

In fact, we could use any letter in place of *x* without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

Note 3

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

that occurs in Definition 2 is called a Riemann sum.

- It is named after the German mathematician Bernhard Riemann (1826–1866).

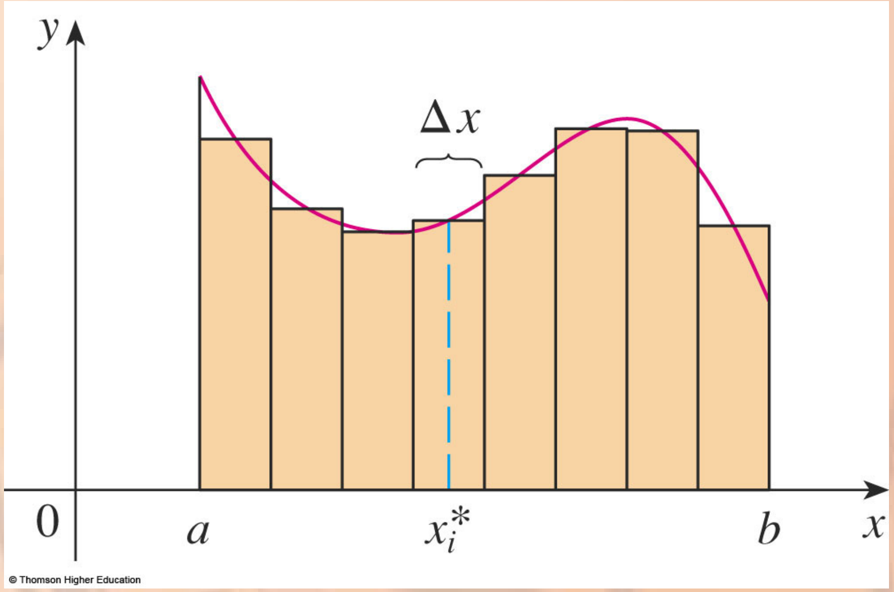
Note 3

So, Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

Note 3

We know that, if *f* happens to be positive, the Riemann sum can be interpreted as:

- A sum of areas of approximating rectangles

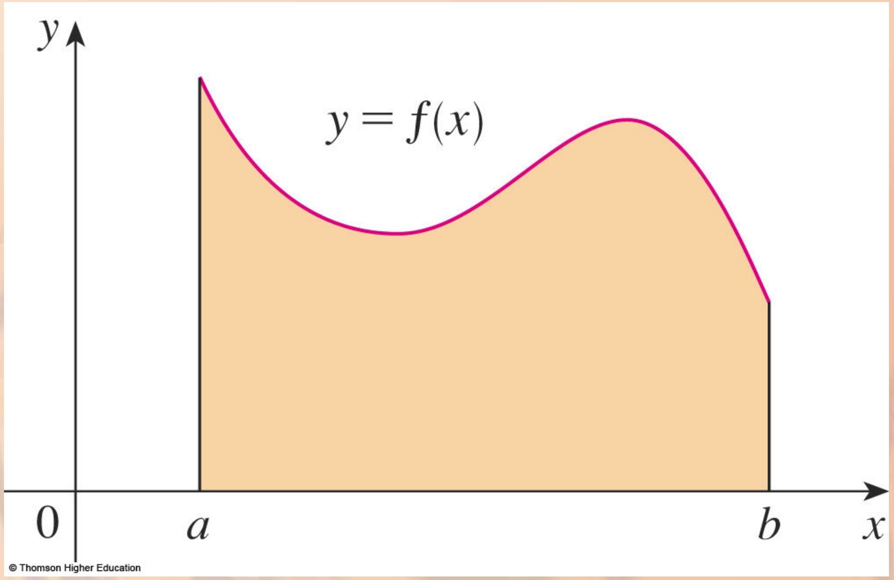


RIEMANN SUM

Note 3

Comparing Definition 2 with the definition of area in Section 5.1, we see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as:

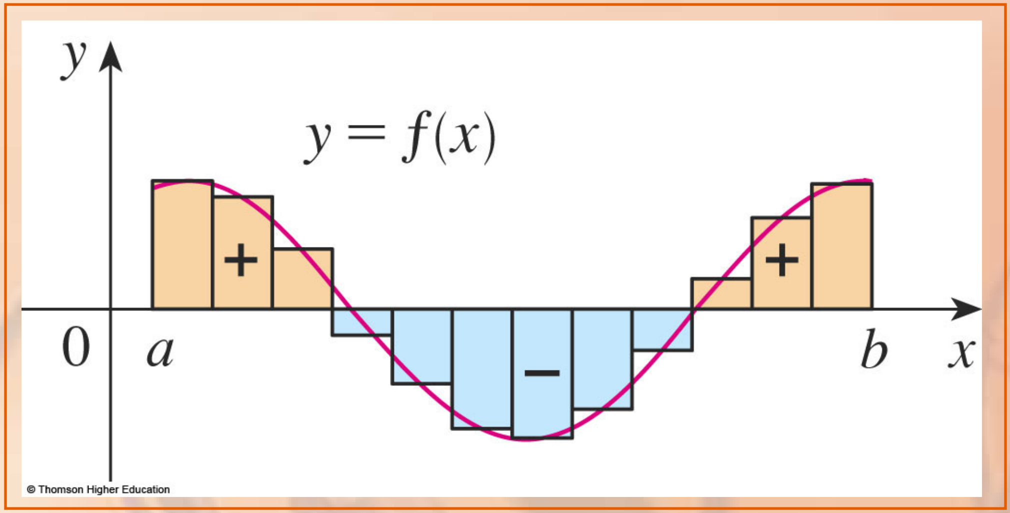
- The area under the curve $y = f(x)$ from a to b



Note 3

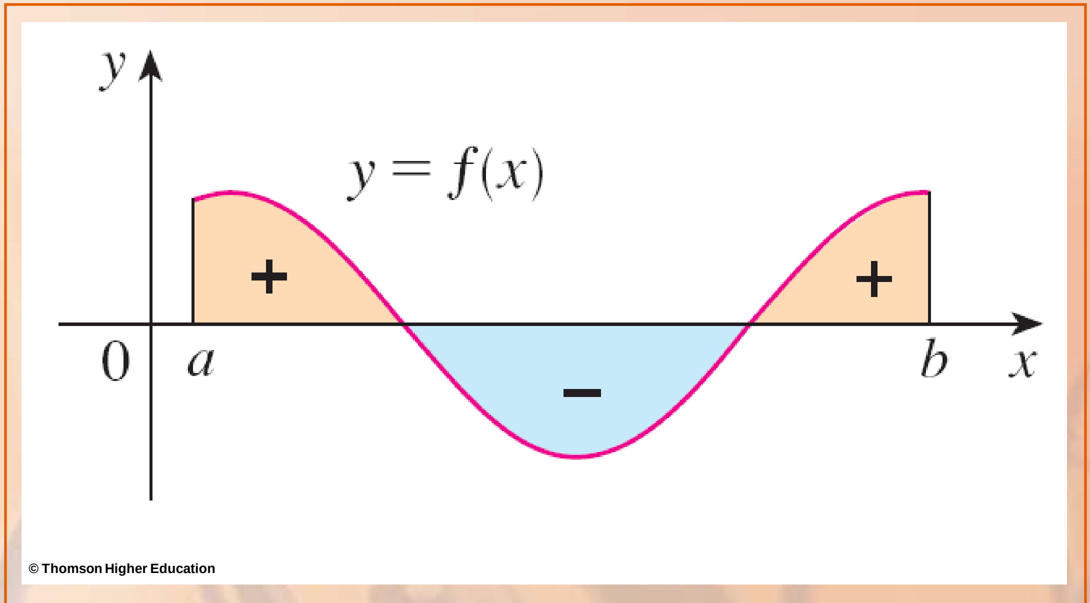
If *f* takes on both positive and negative values, then the Riemann sum is:

- The sum of the areas of the rectangles that lie above the *x*-axis and the negatives of the areas of the rectangles that lie below the *x*-axis
- That is, the areas of the gold rectangles minus the areas of the blue rectangles



Note 3

When we take the limit of such Riemann sums, we get the situation illustrated here.



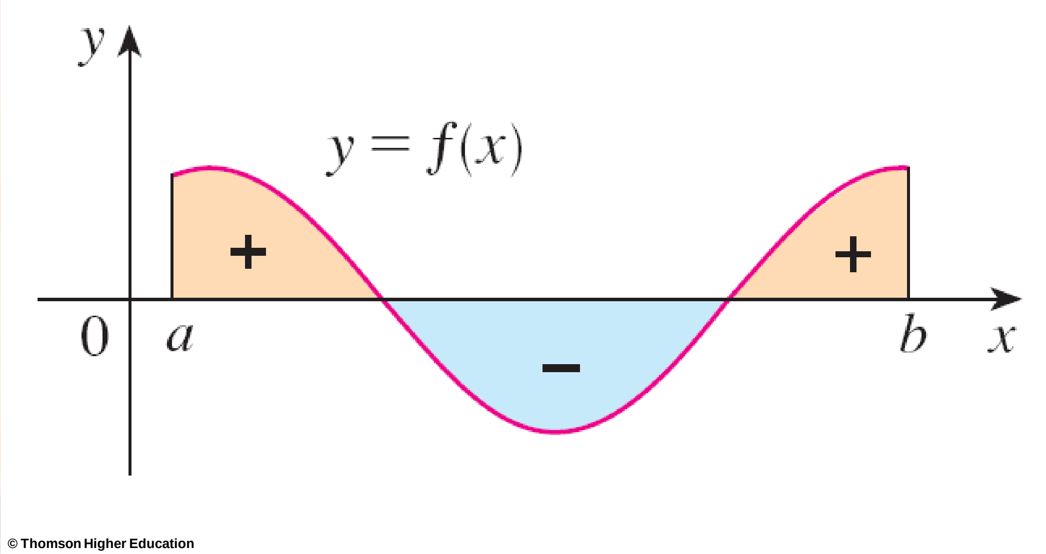
NET AREA

Note 3

A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

- A_1 is the area of the region above the x-axis and below the graph of f.
- A_2 is the area of the region below the x-axis and above the graph of f.



Note 4

Though we have defined $\int_a^b f(x)\,dx$ by dividing [*a*, *b*] into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

- In Exercise 14 in Section 5.1, NASA provided velocity data at times that were not equally spaced.
- We were still able to estimate the distance traveled.

UNEQUAL SUBINTERVALS

Note 4

There are methods for numerical integration that take advantage of unequal subintervals.

UNEQUAL SUBINTERVALS

Note 4

If the subinterval widths are Δx_1, Δx_2, …, Δx_n, we have to ensure that all these widths approach 0 in the limiting process.

- This happens if the largest width, max Δx_i , approaches 0.

Note 4

Thus, in this case, the definition of a definite integral becomes:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

Note 5

We have defined the definite integral for an integrable function.

However, not all functions are integrable.

INTEGRABLE FUNCTIONS

The following theorem shows that the most commonly occurring functions are, in fact, integrable.

- It is proved in more advanced courses.

Theorem 3

If *f* is continuous on [*a*, *b*], or if *f* has only a finite number of jump discontinuities, then *f* is integrable on [*a*, *b*].

That is, the definite integral $\int_a^b f(x)\,dx$ exists.

INTEGRABLE FUNCTIONS

If *f* is integrable on [*a*, *b*], then the limit in Definition 2 exists and gives the same value, no matter how we choose the sample points x_i^*.

INTEGRABLE FUNCTIONS

To simplify the calculation of the integral, we often take the sample points to be right endpoints.

- Then, $x_i^* = x_i$ and the definition of an integral simplifies as follows.

Theorem 4

If *f* is integrable on [*a*, *b*], then

$$\int_a^b f(x)\,dx = \lim_{n_i \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

Example 1

Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x_i$$

as an integral on the interval [0, π].

- Comparing the given limit with the limit in Theorem 4, we see that they will be identical if we choose $f(x) = x^3 + x \sin x$.

We are given that $a = 0$ and $b = \pi$.

So, by Theorem 4, we have:

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x_i = \int_0^{\pi} (x^3 + x \sin x)\, dx$$

DEFINITE INTEGRAL

Later, when we apply the definite integral to physical situations, it will be important to recognize limits of sums as integrals—as we did in Example 1.

DEFINITE INTEGRAL

When Leibniz chose the notation for an integral, he chose the ingredients as reminders of the limiting process.

In general, when we write

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

we replace:

- lim Σ by ∫
- x_i* by x
- Δx by dx

EVALUATING INTEGRALS

When we use a limit to evaluate a definite integral, we need to know how to work with sums.

EVALUATING INTEGRALS

The following three equations give formulas for sums of powers of positive integers.

Equation 5

Equation 5 may be familiar to you from a course in algebra.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Equations 6 & 7

Equations 6 and 7 were discussed in Section 5.1 and are proved in Appendix E.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining formulas are simple rules for working with sigma notation:

$\sum_{i=1}^{n} c = nc$
$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Example 2

a. Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

b. Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

With n = 6,

- The interval width is: $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$
- The right endpoints are:

  $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$,
  $x_4 = 2.0$, $x_5 = 2.5$, $x_6 = 3.0$

Example 2 a

So, the Riemann sum is:

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x$$

$$+ f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x$$

$$= \tfrac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9)$$

$$= -3.9375$$

Example 2 a

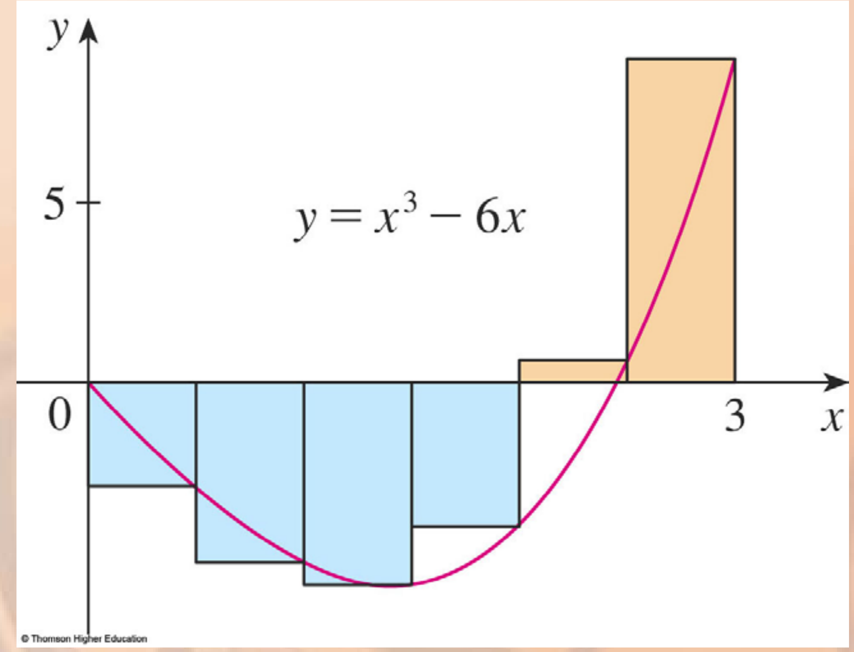
Notice that *f* is not a positive function.

So, the Riemann sum does not represent a sum of areas of rectangles.

EVALUATING INTEGRALS

Example 2 a

However, it does represent the sum of the areas of the gold rectangles (above the *x*-axis) minus the sum of the areas of the blue rectangles (below the *x*-axis).



With *n* subintervals, we have:

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

Thus, $x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $x_3 = 9/n$.

In general, $x_i = 3i/n$.

Example 2 b

Since we are using right endpoints, we can use Theorem 4, as follows.

EVALUATING INTEGRALS

Example 2 b

$$\int_0^3 (x^3 - 6x)\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] \qquad \text{(Eqn. 9 with } c = 3/n\text{)}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

EVALUATING INTEGRALS

Example 2 b

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right] \qquad \text{(Eqns. 11 \& 9)}$$

$$= \lim_{n\to\infty}\left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\} \qquad \text{(Eqns. 7 \& 5)}$$

$$= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$