LINE INTEGRALS OF VECTOR FIELDS

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we found that the work done by a constant force **F** in moving an object from a point *P* to another point in space is:

$$W = \mathbf{F} \cdot \mathbf{D}$$

where **D** = $\overrightarrow{PQ}$ is the displacement vector.

LINE INTEGRALS OF VECTOR FIELDS

Now, suppose that

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

is a continuous force field on $°^{3}$,

such as:

- The gravitational field of Example 4 in Section 12.1
- The electric force field of Example 5 in Section 12.1

LINE INTEGRALS OF VECTOR FIELDS

A force field on ° ^{3}could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.

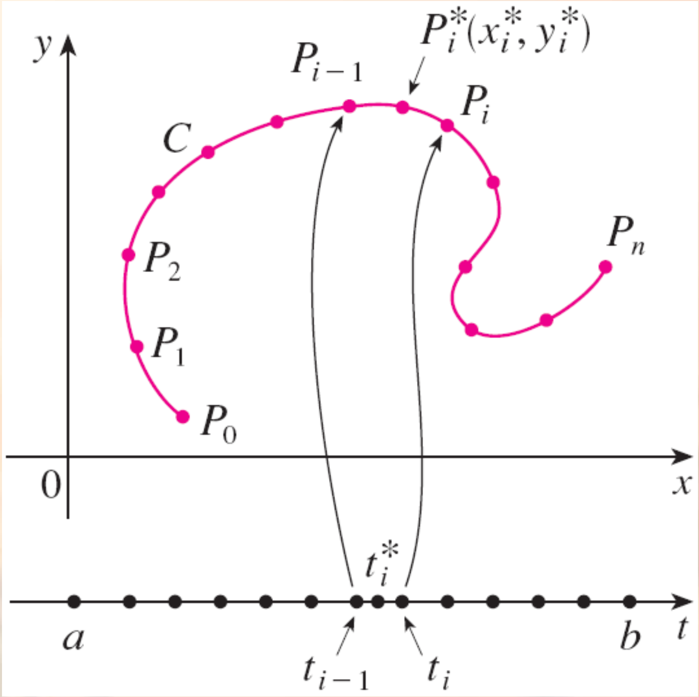
- We wish to compute the work done by this force in moving a particle along a smooth curve C.

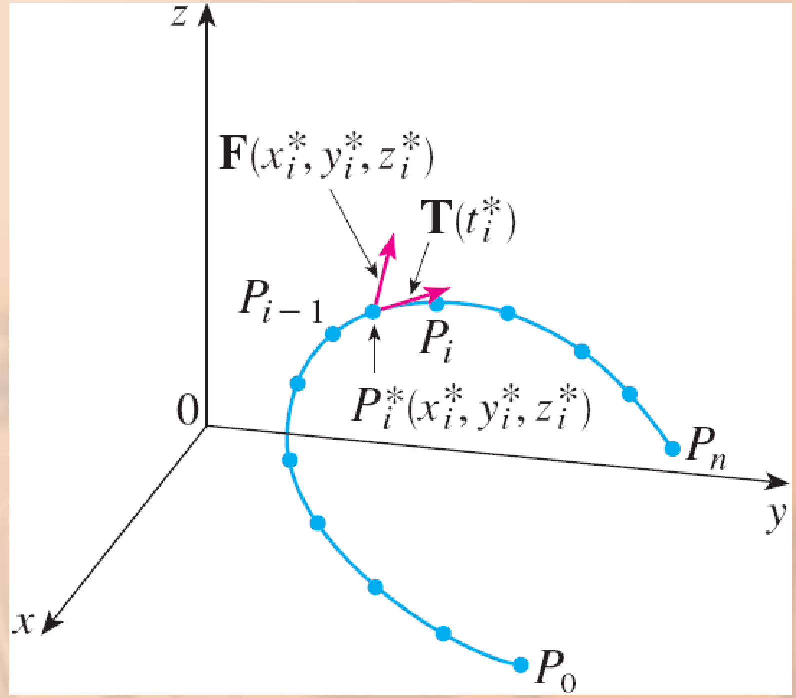
We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width.

LINE INTEGRALS OF VECTOR FIELDS

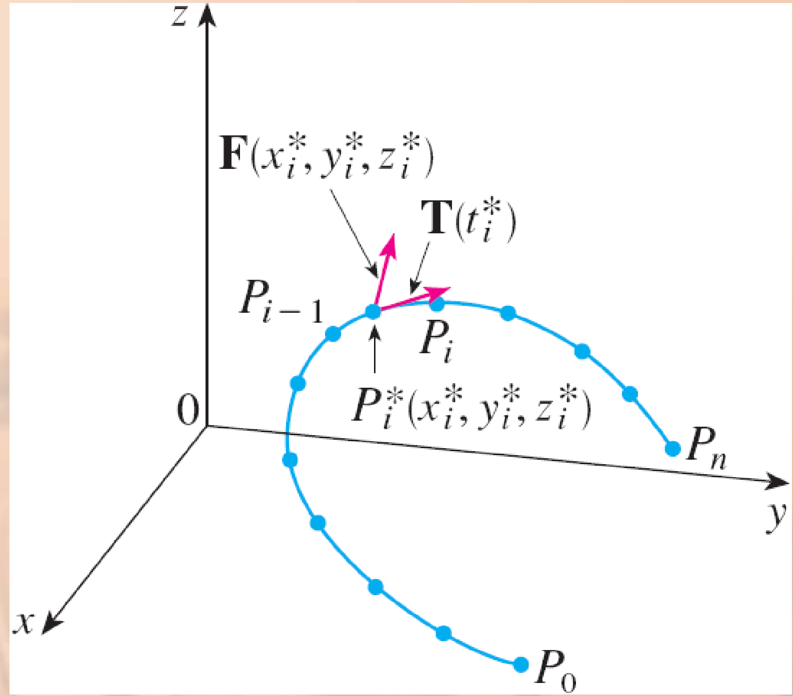
The first figure shows the two-dimensional case.

The second shows the three-dimensional one.



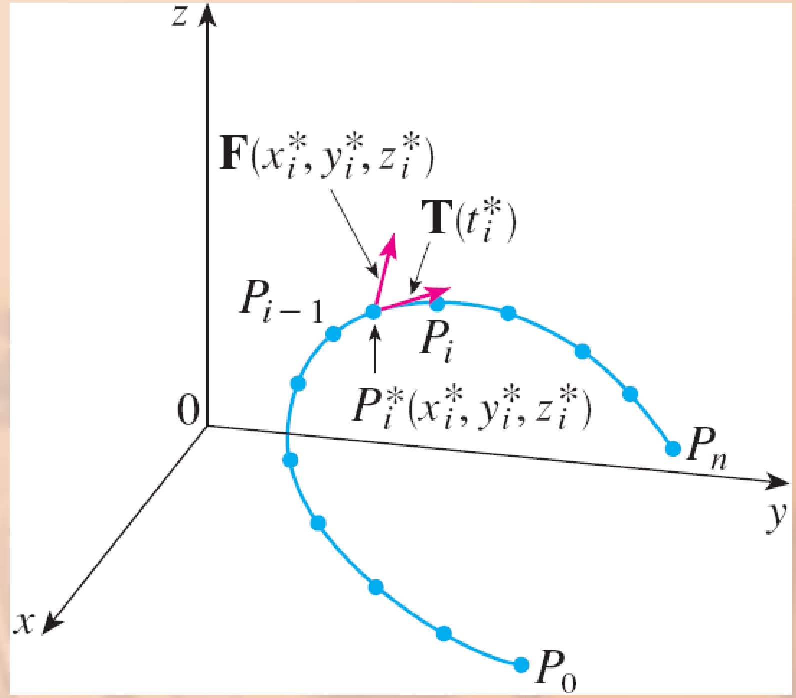


Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the i th subarc corresponding to the parameter value t_i^*.



LINE INTEGRALS OF VECTOR FIELDS

If Δs_i is small, then as the particle moves from $P_{i\text{-}1}$ to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^*.



Thus, the work done by the force **F** in moving the particle P_{i-1} from to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \, \mathbf{T}(t_i^*)]$$
$$= [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \, \Delta s_i$$

The total work done in moving the particle along *C* is approximately

$$\sum_{i=1}^{n}\left[\mathbf{F}(x_i^*, y_i^*, z_i^*)\cdot\mathbf{T}(x_i^*, y_i^*, z_i^*)\right]\Delta s_i$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on *C*.

Intuitively, we see that these approximations ought to become better as *n* becomes larger.

Equation 12

Thus, we define the work *W* done by the force field **F** as the limit of the Riemann sums in Formula 11, namely,

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

- This says that work is the line integral with respect to arc length of the tangential component of the force.

If the curve *C* is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

then

$$\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$$

So, using Equation 9, we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

This integral is often abbreviated

as $\int_C \mathbf{F} \cdot d\mathbf{r}$

and occurs in other areas of physics as well.

- Thus, we make the following definition for the line integral of any continuous vector field.

Definition 13

Let **F** be a continuous vector field defined on a smooth curve *C* given by a vector function $\mathbf{r}(t)$, $a \le t \le b$.

Then, the line integral of **F** along *C* is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

When using Definition 13, remember $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for

$$\mathbf{F}(x(t), y(t), z(t))$$

- So, we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting
  $$x = x(t),\ y = y(t), \text{ and } z = z(t)$$
  in the expression for $\mathbf{F}(x, y, z)$.

- Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

Find the work done by the force field

$$\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$$

in moving a particle along the quarter-circle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \le t \le \pi/2$$

Since $x = \cos t$ and $y = \sin t$,

we have:

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t \, \mathbf{i} - \cos t \sin t \, \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t \, \mathbf{i} + \cos t \, \mathbf{j}$$

Therefore, the work done is:

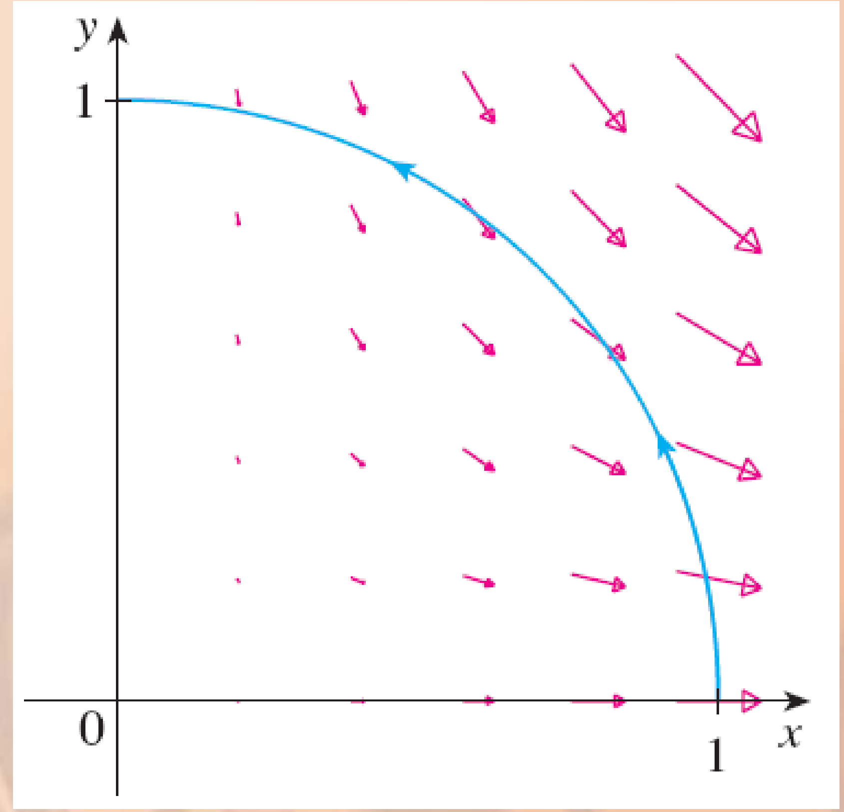
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{\pi/2} \left(-2\cos^2 t \sin t\right) dt$$

$$= 2\frac{\cos^3 t}{3}\Big]_0^{\pi/2} = -\frac{2}{3}$$

The figure shows the force field and the curve in Example 7.

- The work done is negative because the field impedes movement along the curve.



Note

Although $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that:

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

- This is because the unit tangent vector **T** is replaced by its negative when *C* is replaced by –*C*.

Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where:

- $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$
- C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leq t \leq 1$$

We have:

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

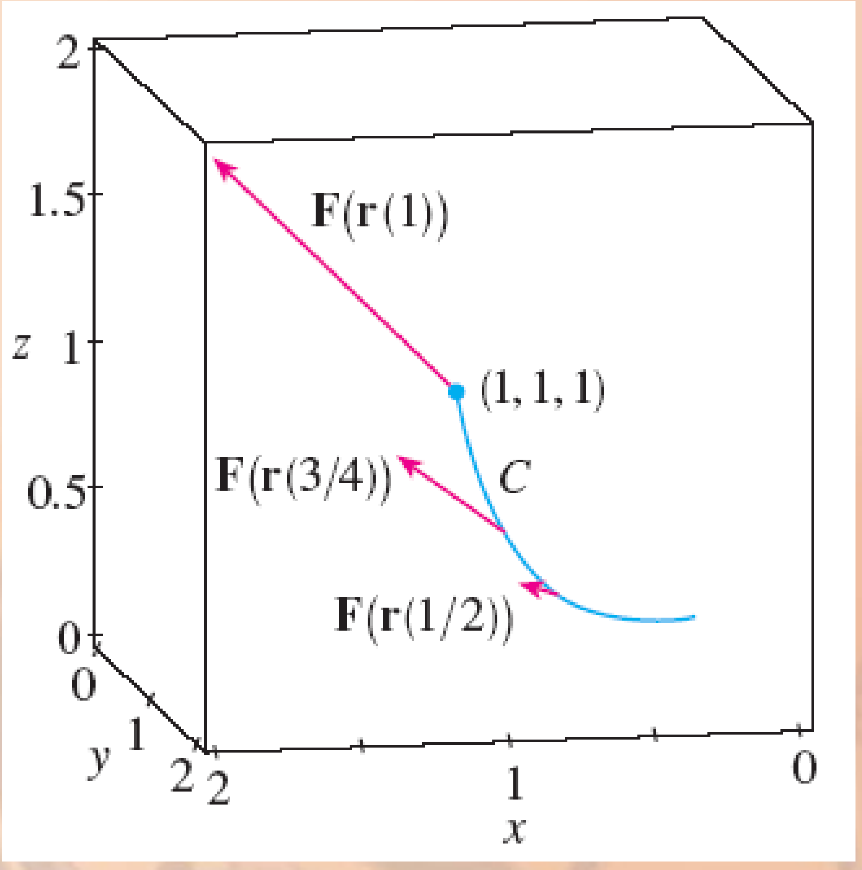
Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t) \cdot \mathbf{r}'(t))\, dt$$

$$= \int_0^1 \left(t^3 + 5t^6\right) dt$$

$$= \frac{t^4}{4} + \frac{5t^7}{7}\Big]_0^1 = \frac{27}{28}$$

The figure shows the twisted cubic in Example 8 and some typical vectors acting at three points on *C*.



Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields.

VECTOR & SCALAR FIELDS

Suppose the vector field **F** on ° ^{3} is given in component form by:

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

- We use Definition 13 to compute its line integral along *C*, as follows.

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k})\, dt$$

$$= \int_a^b \begin{bmatrix} P(x(t), y(t), z(t))\, x'(t) \\ +Q(x(t), y(t), z(t))\, y'(t) \\ +R(x(t), y(t), z(t))\, z'(t) \end{bmatrix}$$

However, that last integral is precisely the line integral in Formula 10.

Hence, we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

where **F** = *P* **i** + *Q* **j** + *R* **k**

For example, the integral

$$\int_C y\,dx + z\,dy + x\,dz$$

in Example 6 could be expressed as

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$