Directional Derivatives

In this section, we will learn how to find:

The rate of changes of a function of

two or more variables in any direction.

INTRODUCTION

This weather map shows a contour map of the temperature function $T(x, y)$ for:

- The states of California and Nevada at 3:00 PM on a day in October.

INTRODUCTION

The level curves, or isothermals, join locations with the same temperature.

INTRODUCTION

The partial derivative T_x is the rate of change of temperature with respect to distance if we travel east from Reno.

- T_y is the rate of change if we travel north.

INTRODUCTION

However, what if we want to know the rate of change when we travel southeast (toward Las Vegas), or in some other direction?

DIRECTIONAL DERIVATIVE

In this section, we introduce a type of derivative, called a directional derivative, that enables us to find:

- The rate of change of a function of two or more variables in any direction.

Recall that, if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as:

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

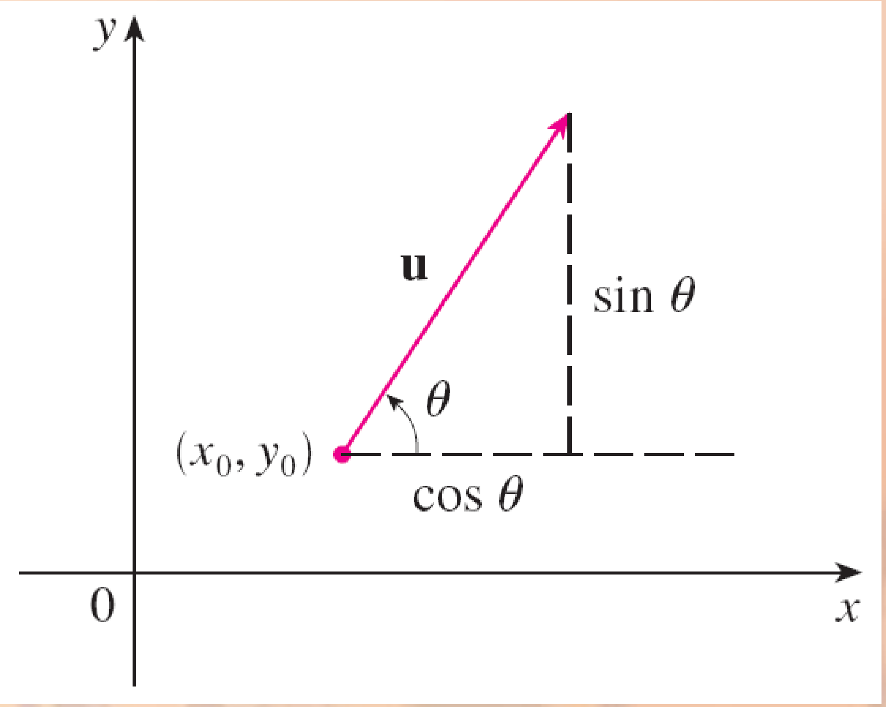
$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

 Equations 1

They represent the rates of change of z in the x- and y-directions—that is, in the directions of the unit vectors **i** and **j**.

DIRECTIONAL DERIVATIVES

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$.

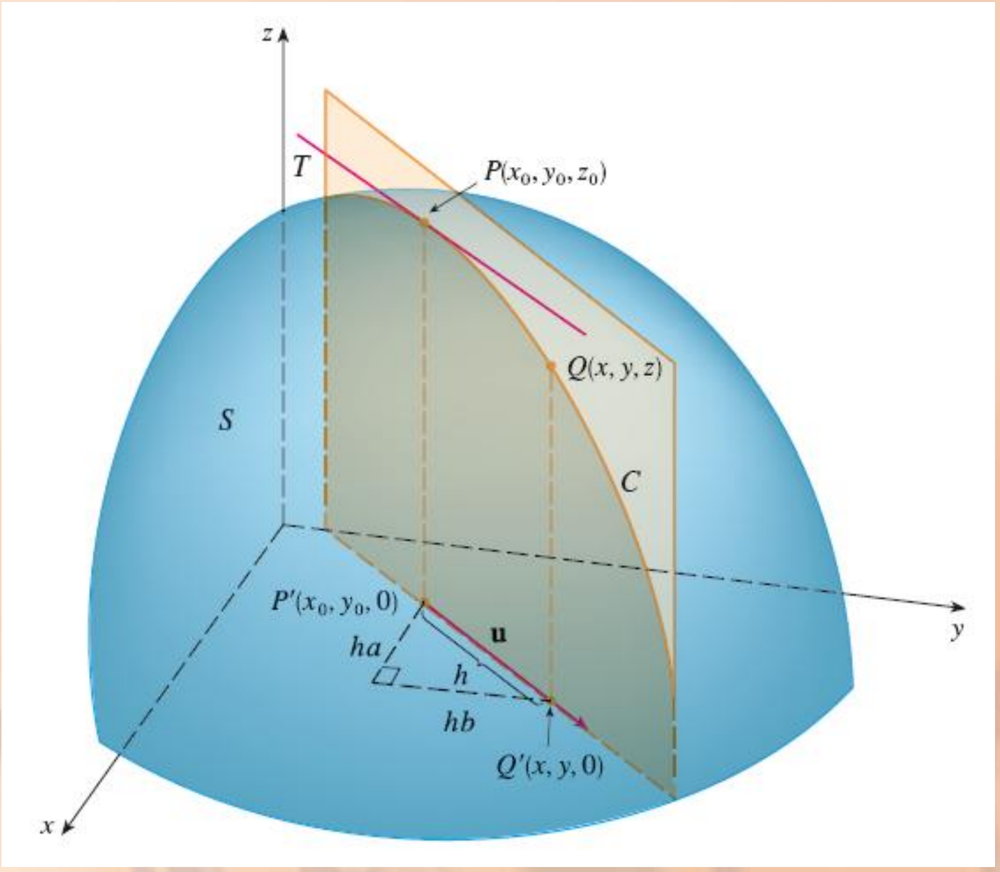


To do this, we consider the surface S with equation $z = f(x, y)$ [the graph of f] and we let $z_0 = f(x_0, y_0)$.

- Then, the point $P(x_0, y_0, z_0)$ lies on S.

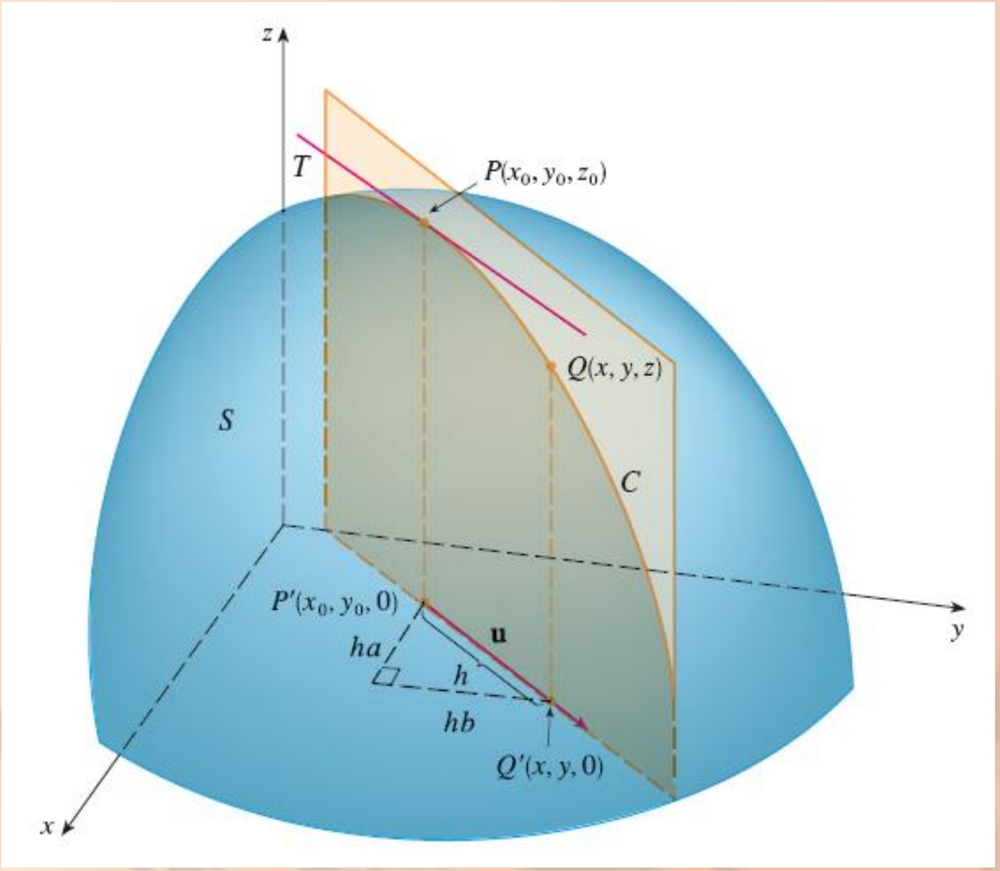
DIRECTIONAL DERIVATIVES

The vertical plane that passes through *P* in the direction of **u** intersects *S* in a curve *C*.



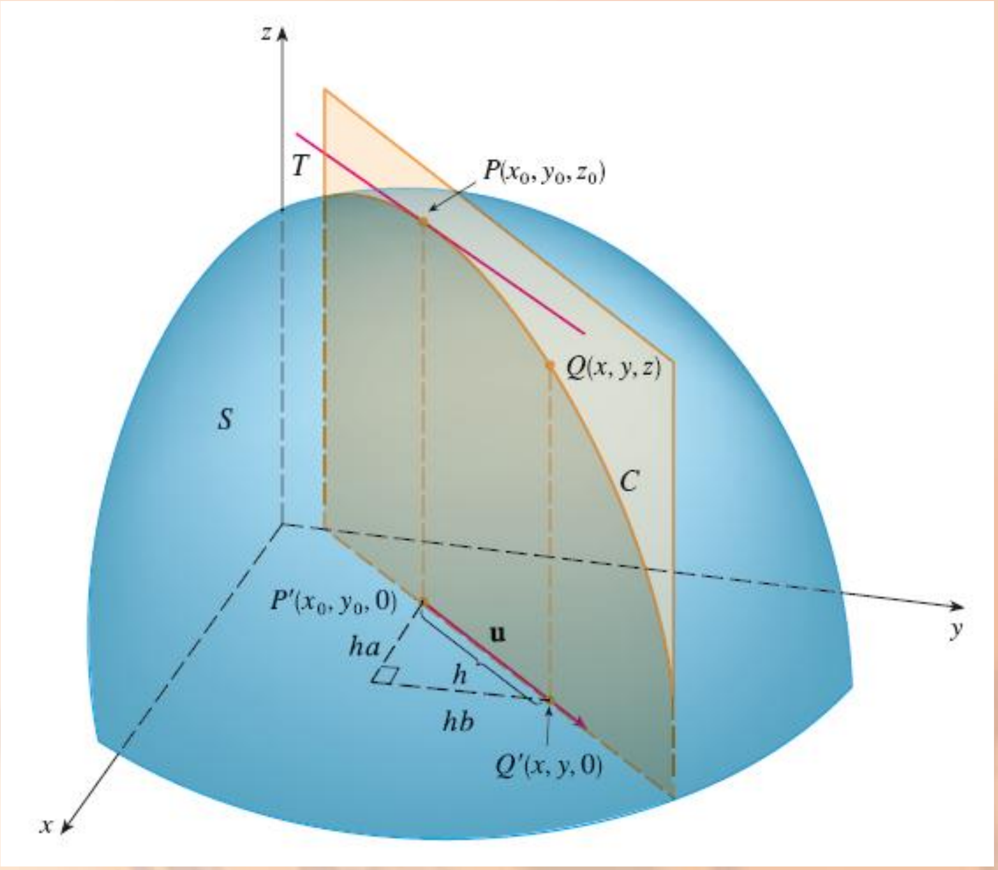
DIRECTIONAL DERIVATIVES

The slope of the tangent line *T* to *C* at the point *P* is the rate of change of *z* in the direction of **u**.



Now, let:

- *Q*(*x*, *y*, *z*) be another point on *C*.
- *P'*, *Q'* be the projections of *P*, *Q* on the *xy*-plane.



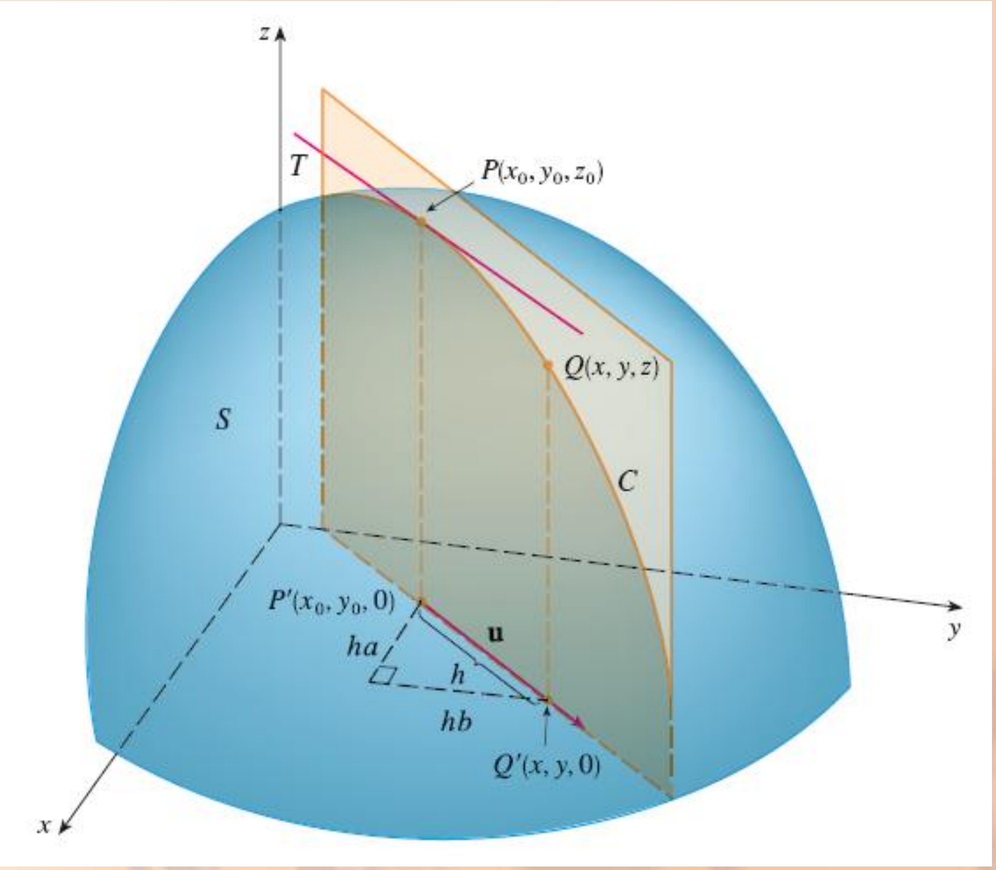
Then, the vector $\overrightarrow{P'Q'}$ is parallel to **u**.

So,

$$\overrightarrow{P'Q'} = h\mathbf{u}$$

$$= \langle ha, hb \rangle$$

for some scalar *h*.



Therefore,

$$x - x_0 = ha$$

$$y - y_0 = hb$$

So,

$$x = x_0 + ha$$

$$y = y_0 + hb$$

$$\frac{\Delta z}{h} = \frac{z - z_0}{h}$$

$$= \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

DIRECTIONAL DERIVATIVE

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of **u.**

- This is called the directional derivative of f in the direction of **u**.

DIRECTIONAL DERIVATIVE — Definition 2

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is:

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Comparing Definition 2 with Equations 1, we see that:

- If **u** = **i** = ⟨1, 0⟩, then $D_i f = f_x$.
- If **u** = **j** = ⟨0, 1⟩, then $D_j f = f_y$.

DIRECTIONAL DERIVATIVES

In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

DIRECTIONAL DERIVATIVES Example 1

Use this weather map to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

The unit vector directed toward the southeast is:

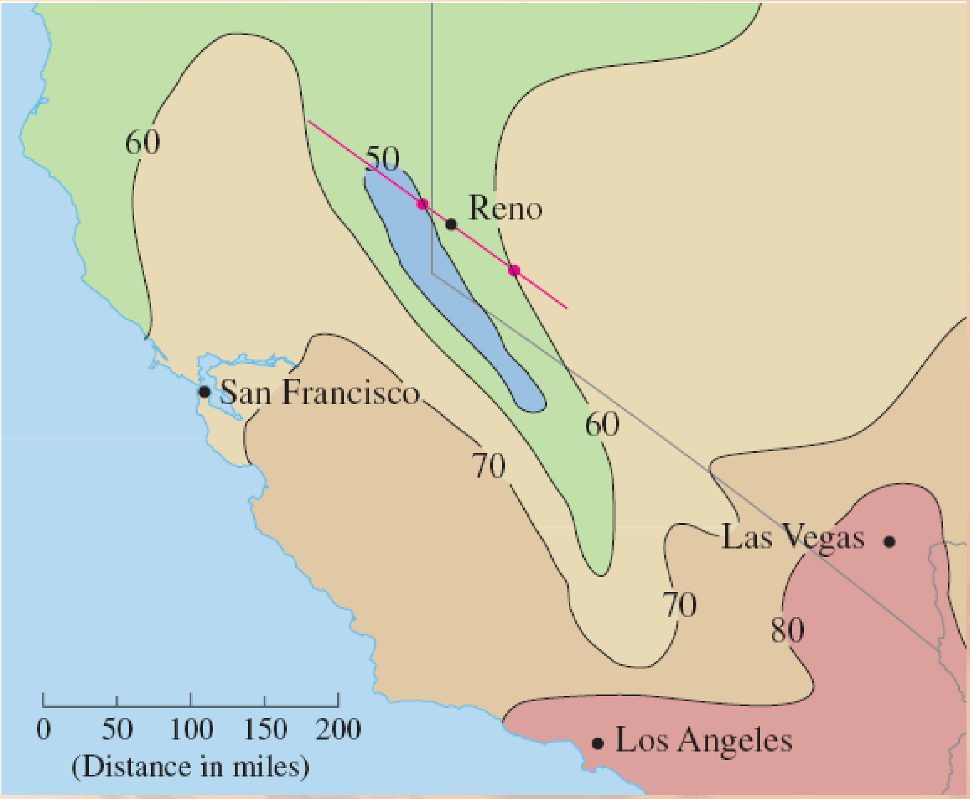
$$\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$$

- However, we won't need to use this expression.

We start by drawing a line through Reno toward the southeast.

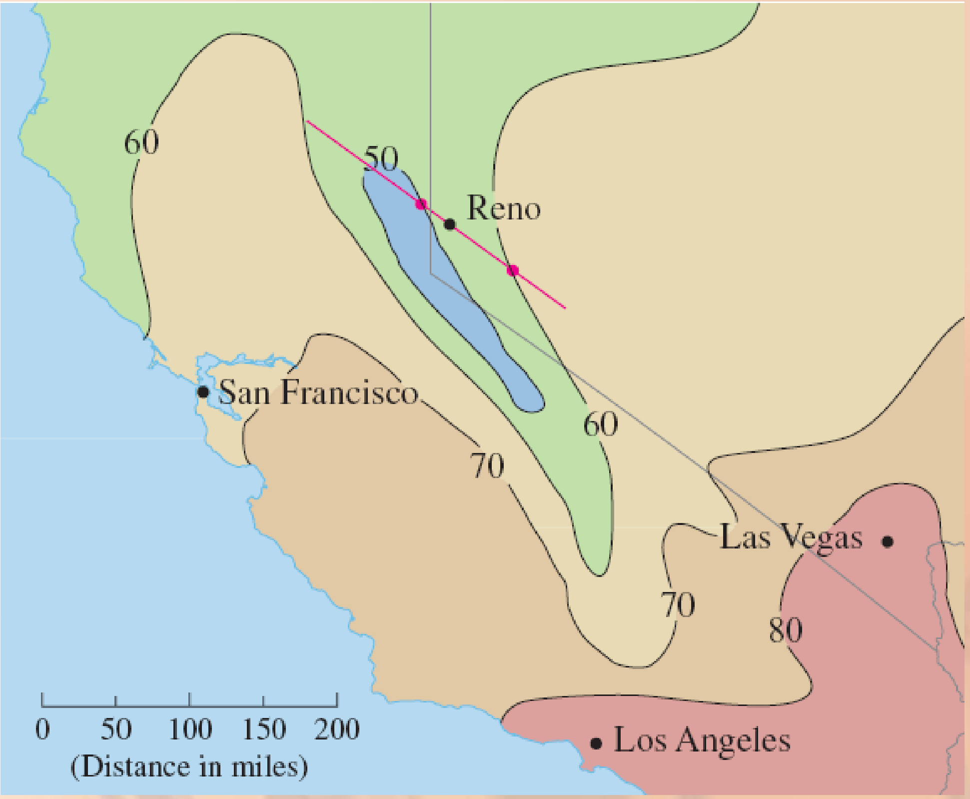
We approximate the directional derivative $D_{\mathbf{u}}T$ by:

- The average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and $T = 60$.

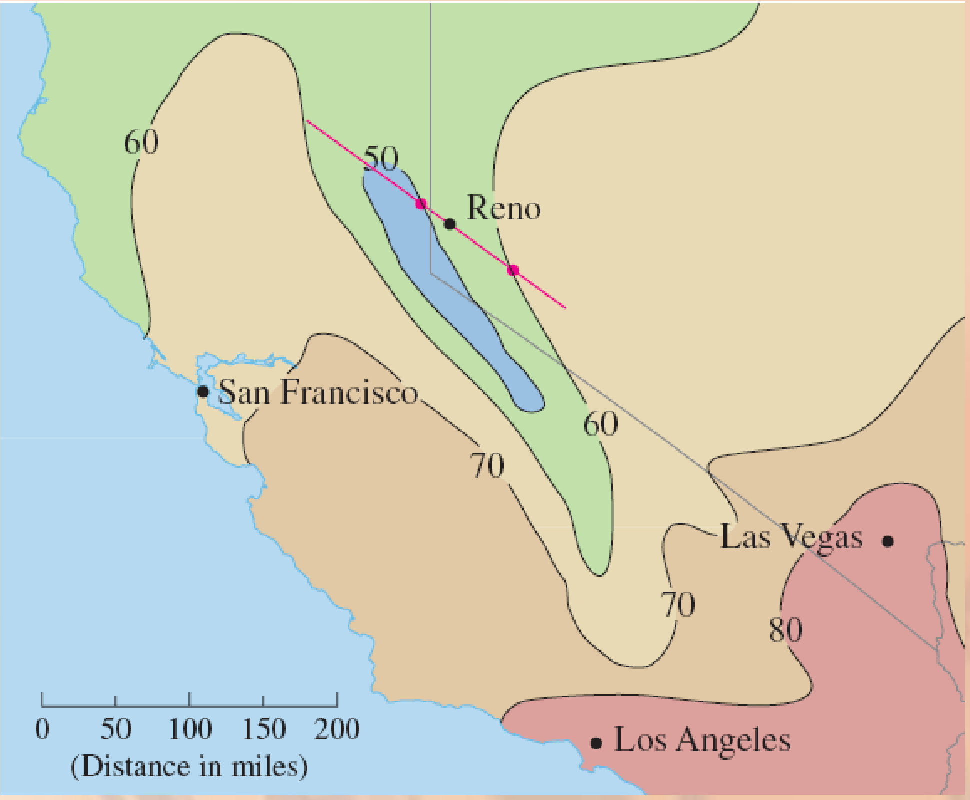


The temperature at the point southeast of Reno is T = 60°F.

The temperature at the point northwest of Reno is T = 50°F.



The distance between these points looks to be about 75 miles.



So, the rate of change of the temperature in the southeasterly direction is:

$$D_{\mathbf{u}}T \approx \frac{60-50}{75}$$

$$= \frac{10}{75}$$

$$\approx 0.13^{\circ}\ \text{F/mi}$$

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

Theorem 3

If *f* is a differentiable function of *x* and *y*, then *f* has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\,a + f_y(x, y)\,b$$

If we define a function *g* of the single variable *h* by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have the following equation.

Proof—Equation 4

$$g'(0)$$

$$= \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write:

$$g(h) = f(x, y)$$

where:

- $x = x_0 + ha$
- $y = y_0 + hb$

Hence, the Chain Rule (Theorem 2 in Section 10.5) gives:

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$= f_x(x,y)a + f_y(x,y)b$$

Proof—Equation 5

If we now put $h = 0$,

then

$$x = x_0$$

$$y = y_0$$

and

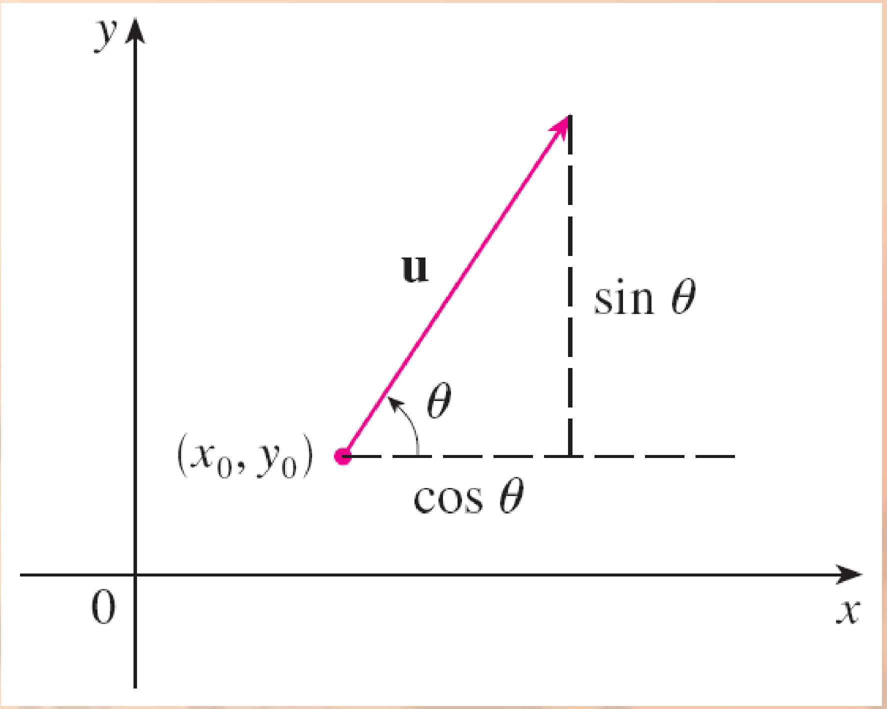
$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Comparing Equations 4 and 5,

we see that:

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

Suppose the unit vector **u** makes an angle *θ* with the positive *x*-axis, as shown.



Equation 6

Then, we can write

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$$

and the formula in Theorem 3 becomes:

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Example 2

Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if:

- $f(x, y) = x^3 - 3xy + 4y^2$
- **u** is the unit vector given by angle $\theta = \pi/6$

What is $D_{\mathbf{u}}f(1, 2)$?

Formula 6 gives:

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2}$$

$$= \tfrac{1}{2}\left[3\sqrt{3}x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right]$$

Therefore,

$$D_{\mathbf{u}} f(1,2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right]$$

$$= \frac{13 - 3\sqrt{3}}{2}$$

DIRECTIONAL DERIVATIVES

The directional derivative $D_{\mathbf{u}} f(1, 2)$ in Example 2 represents the rate of change of z in the direction of **u**.

This is the slope of the tangent line to the curve of intersection of the surface

$$z = x^3 - 3xy + 4y^2$$

and the vertical plane through (1, 2, 0) in the direction of **u** shown here.