CHAPTER 12 MULTIPLE INTEGRALS

CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

CHANGE OF VARIABLES IN SINGLE INTEGRALS

- In one-dimensional calculus. we often use a change of variable (a substitution) to simplify an integral.

TRANSFORMATION

- More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by

$$x = g(u, v) \qquad y = h(u, v)$$

  - We sometimes write these as

$$x = x(u, v) \qquad y = y(u, v)$$

C^1 TRANSFORMATION

- We usually assume that T is a $\boldsymbol{C^1}$ **transformation**.
  - This means that g and h have continuous first-order partial derivatives.

TRANSFORMATION

- A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$.

IMAGE & ONE-TO-ONE TRANSFORMATION

- If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1).
- If no two points have the same image, T is called **one-to-one**.

CHANGE OF VARIABLES

- Figure 1 shows the effect of a transformation *T* on a region *S* in the *uv*-plane.
  - *T* transforms *S* into a region *R* in the *xy*-plane called the **image of *S***, consisting of the images of all points in *S*.

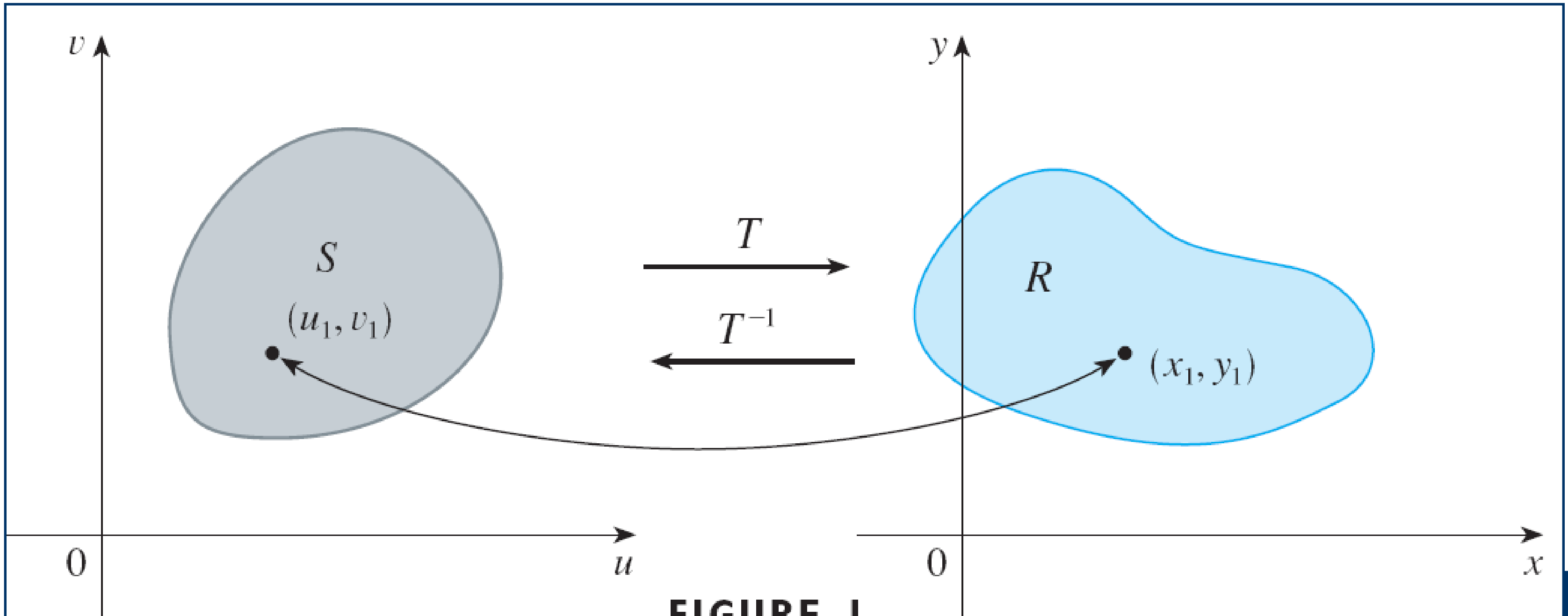


FIGURE 1

INVERSE TRANSFORMATION

- If T is a one-to-one transformation, it has an **inverse transformation** T^{-1} from the *xy*–plane to the *uv*-plane.

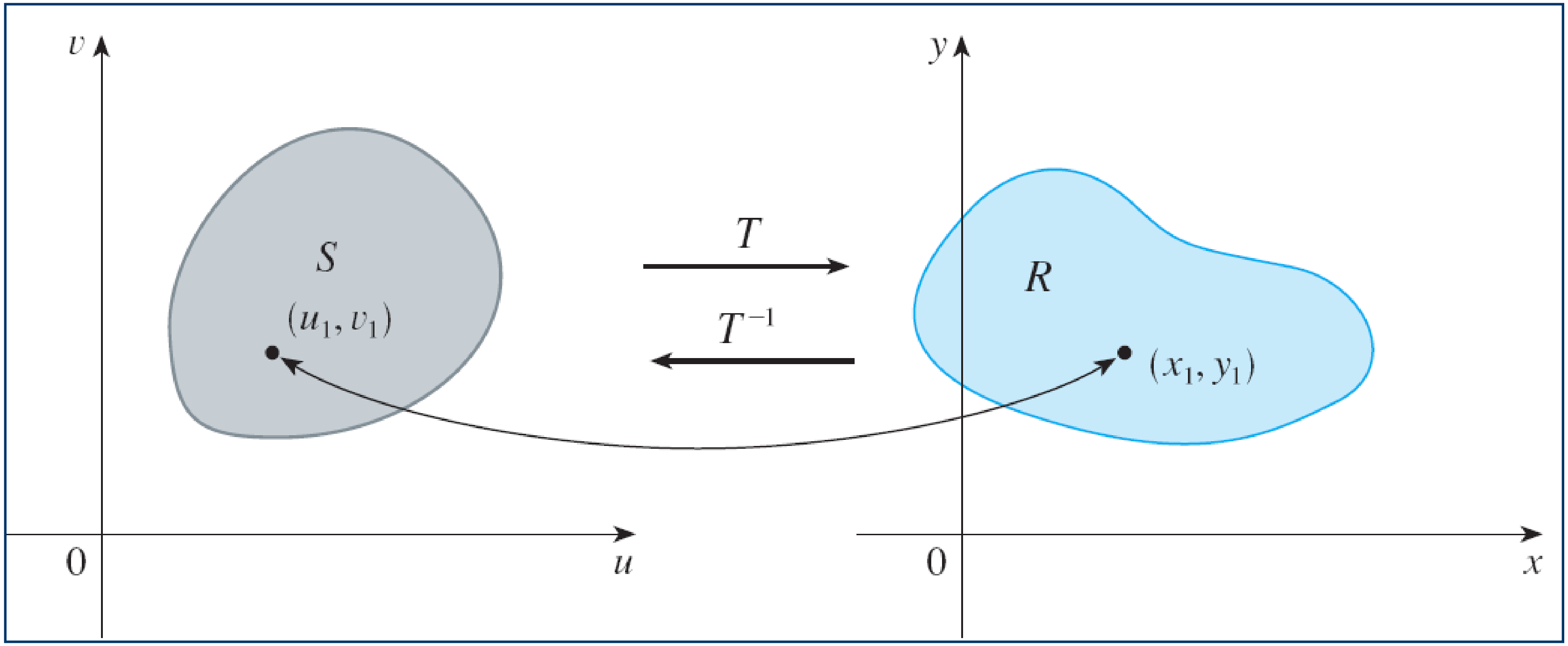


FIGURE 1

INVERSE TRANSFORMATION

- Then, it may be possible to solve Equations 3 for u and v in terms of x and y :

$$u = G(x, y)$$

$$v = H(x, y)$$

Example 1

- A transformation is defined by:

$$x = u^2 - v^2$$
$$y = 2uv$$

- Find the image of the square

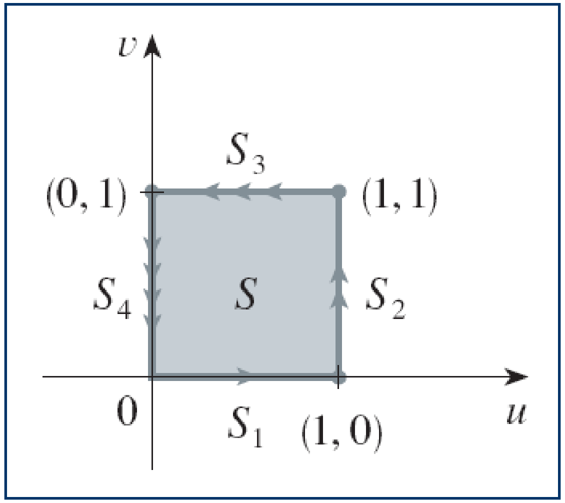
$$S = \{(u, v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$$

Example 1 SOLUTION

- The transformation maps the boundary of S into the boundary of the image.
  - So, we begin by finding the images of the sides of S.
- The first side, S_1, is given by:

$$v = 0 \ (0 \leq u \leq 1)$$

  - See Figure 2.



Example 1 SOLUTION

- From the given equations, we have:

$$x = u^2,\ y = 0, \text{ and so } 0 \le x \le 1.$$

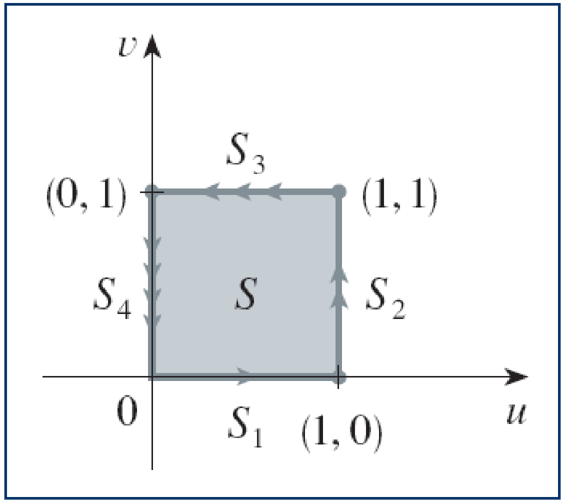
  - Thus, S_1 is mapped into the line segment from (0, 0) to (1, 0) in the *xy*-plane.

- The second side, S_2, is:

$$u = 1\ (0 \le v \le 1)$$

  - Putting $u = 1$ in the given equations, we get:

$$x = 1 - v^2$$
$$y = 2v$$



Example 1 SOLUTION

- Eliminating v, we obtain:

$$x = 1 - \frac{y^2}{4} \qquad 0 \le x \le 1$$

which is part of a parabola.

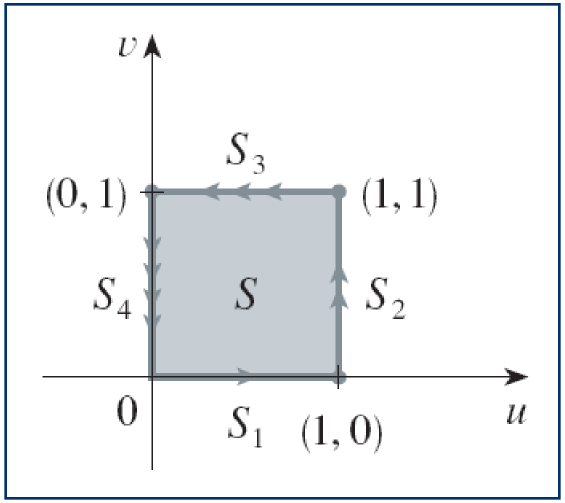
- Similarly, S_3 is given by:

$$v = 1 \ (0 \le u \le 1)$$

- Its image is the parabolic arc

$$x = \frac{y^2}{4} - 1$$

$$(-1 \le x \le 0)$$



Example 1 SOLUTION

- Finally, S_4 is given by:

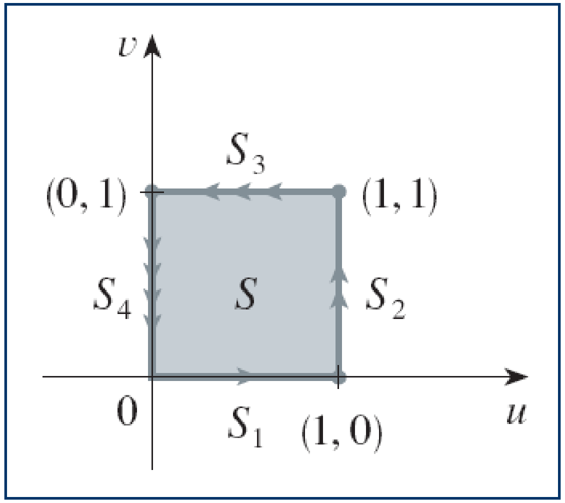
$$u = 0 (0 \le v \le 1)$$

- Its image is:

$$x = -v^2, y = 0$$

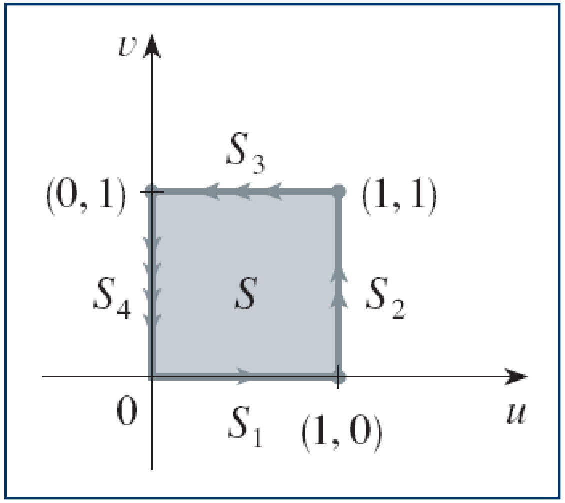
that is,

$$-1 \le x \le 0$$



Example 1 SOLUTION

- Notice that as, we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.



Example 1 SOLUTION

- The image of S is the region R (shown in Figure 2) bounded by:
  - The x-axis.
  - The parabolas given by Equations 4 and 5.

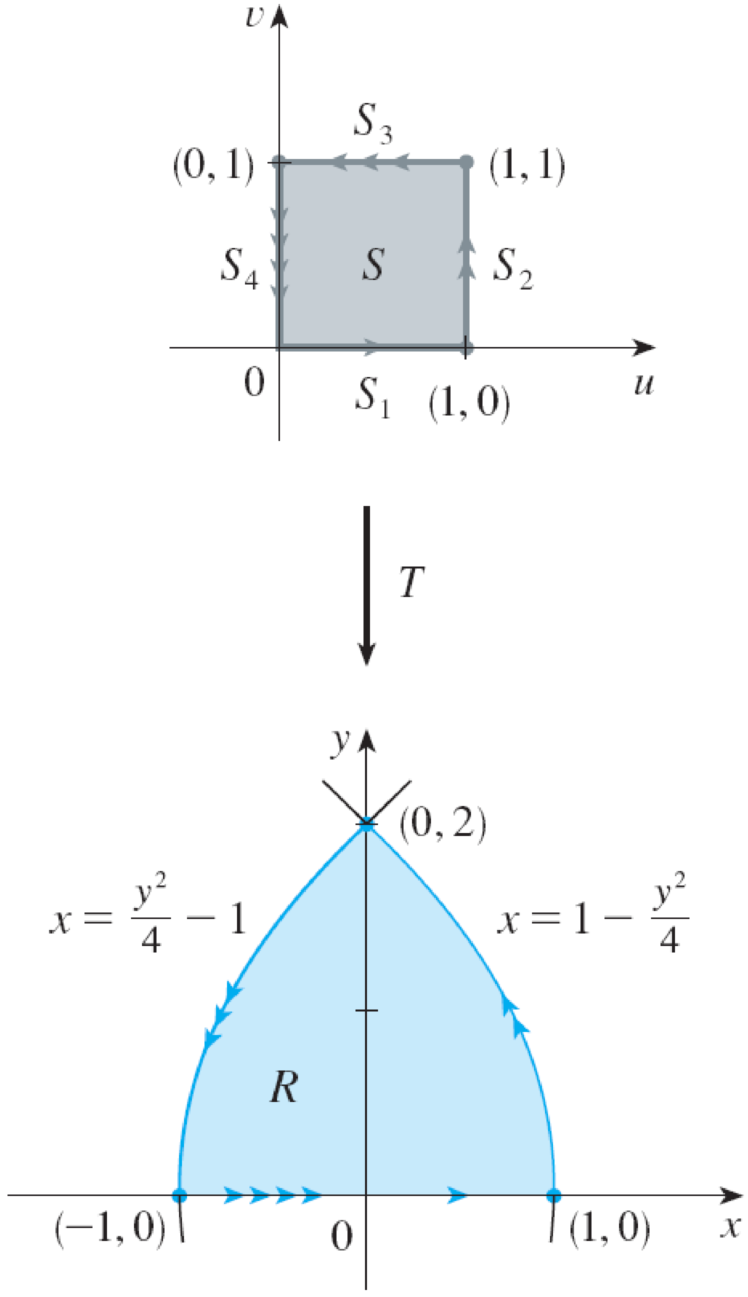
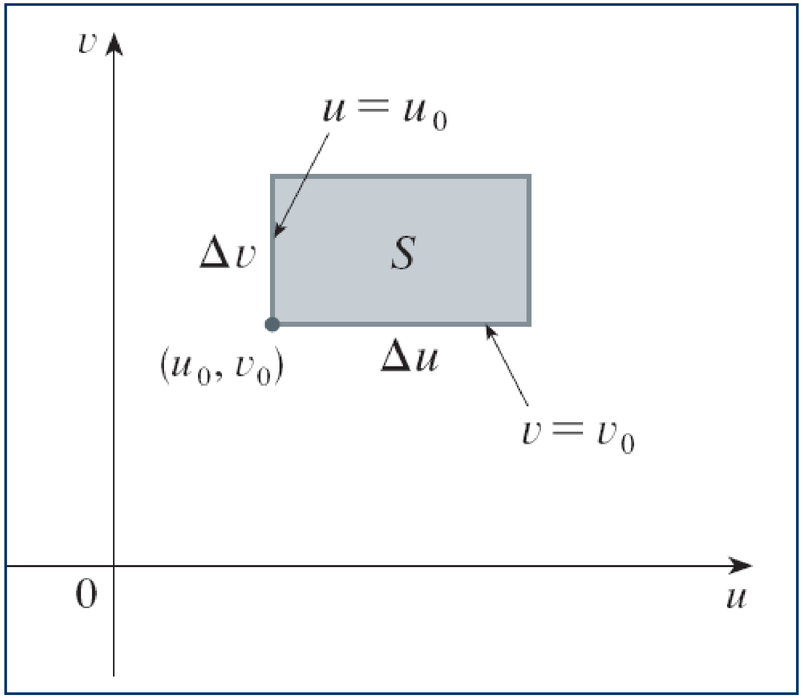


FIGURE 2

DOUBLE INTEGRALS

- Now, let's see how a change of variables affects a double integral.
- We start with a small rectangle *S* in the *uv*-plane whose:
  - Lower left corner is the point (u_0, v_0).
  - Dimensions are Δu and Δv.
  - See Figure 3.



DOUBLE INTEGRALS

- The image of S is a region R in the xy-plane, one of whose boundary points is:

$$(x_0, y_0) = T(u_0, v_0)$$

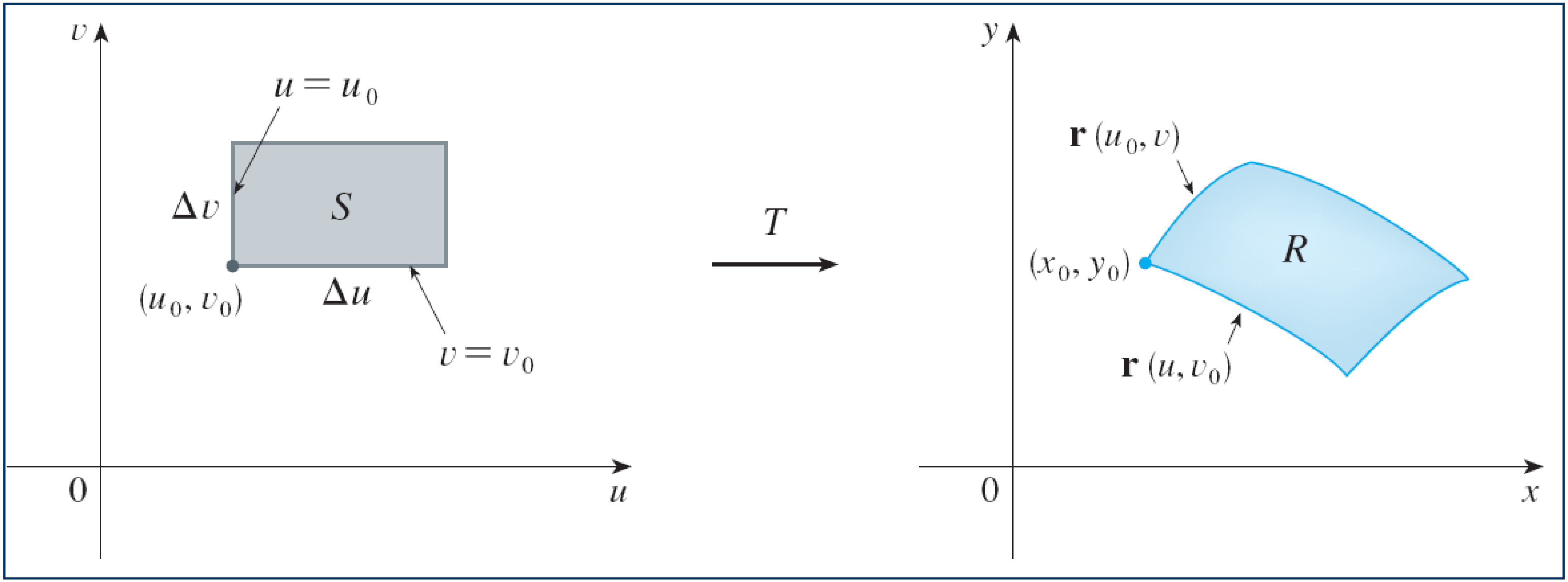


FIGURE 3

- The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

is the position vector of the image of the point (u, v).

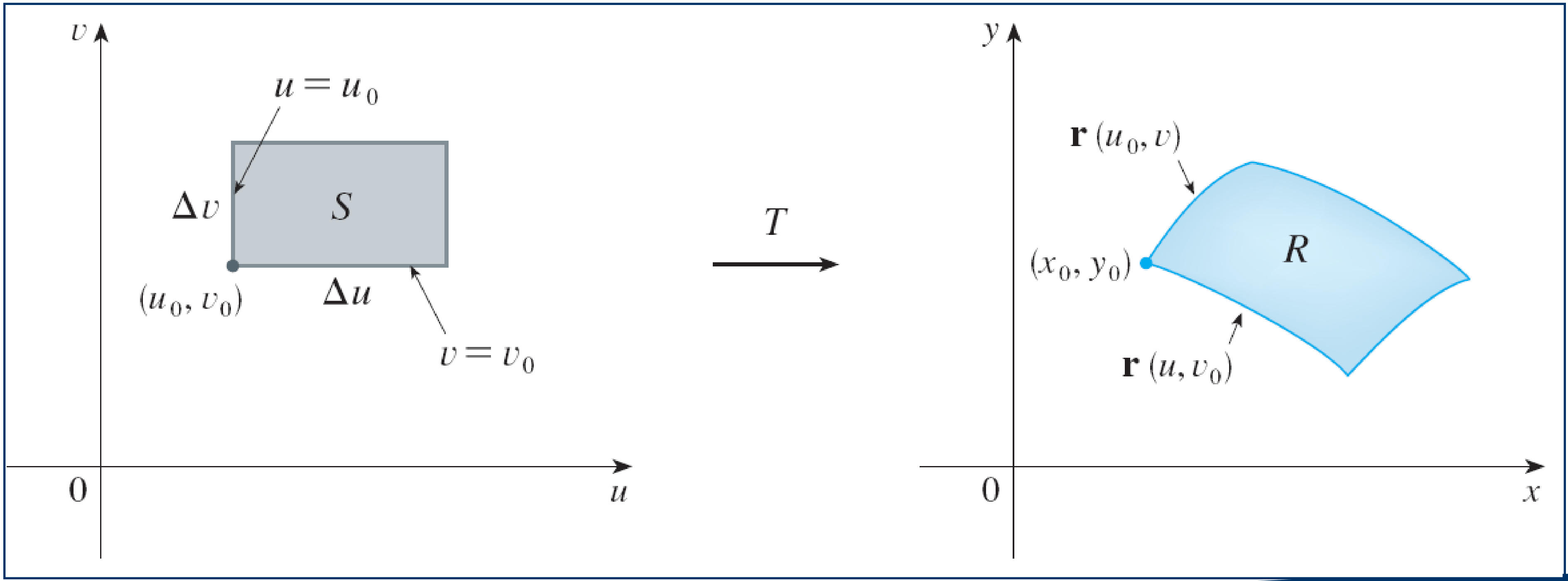


FIGURE 3

- The equation of the lower side of S is:

$$v = v_0$$

  - Its image curve is given by the vector function $\mathbf{r}(u, v_0)$.

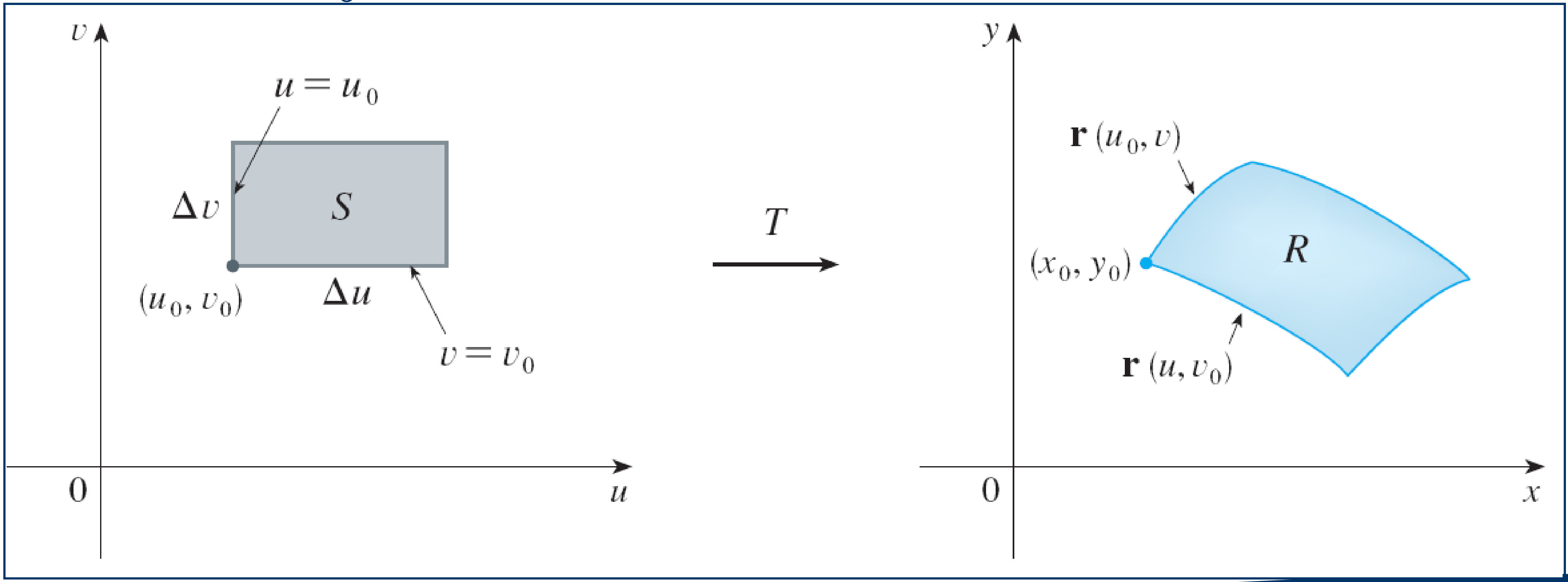


FIGURE 3

- The tangent vector at (x_0, y_0) to this image curve is:

$$\mathbf{r}_u = g_u(u_0, v_0)\,\mathbf{i} + h_u(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}$$

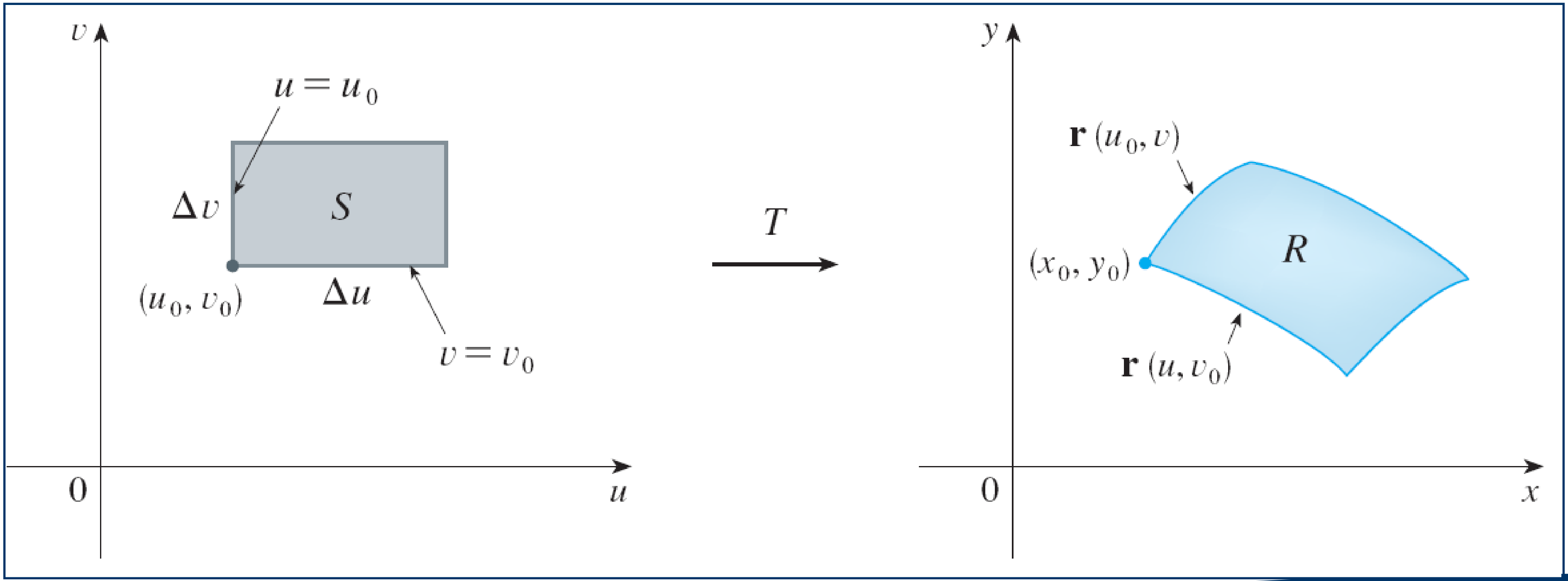


FIGURE 3

❖ Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S $(u = u_0)$ is:

$$\mathbf{r}_v = g_v(u_0, v_0)\,\mathbf{i} + h_v(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

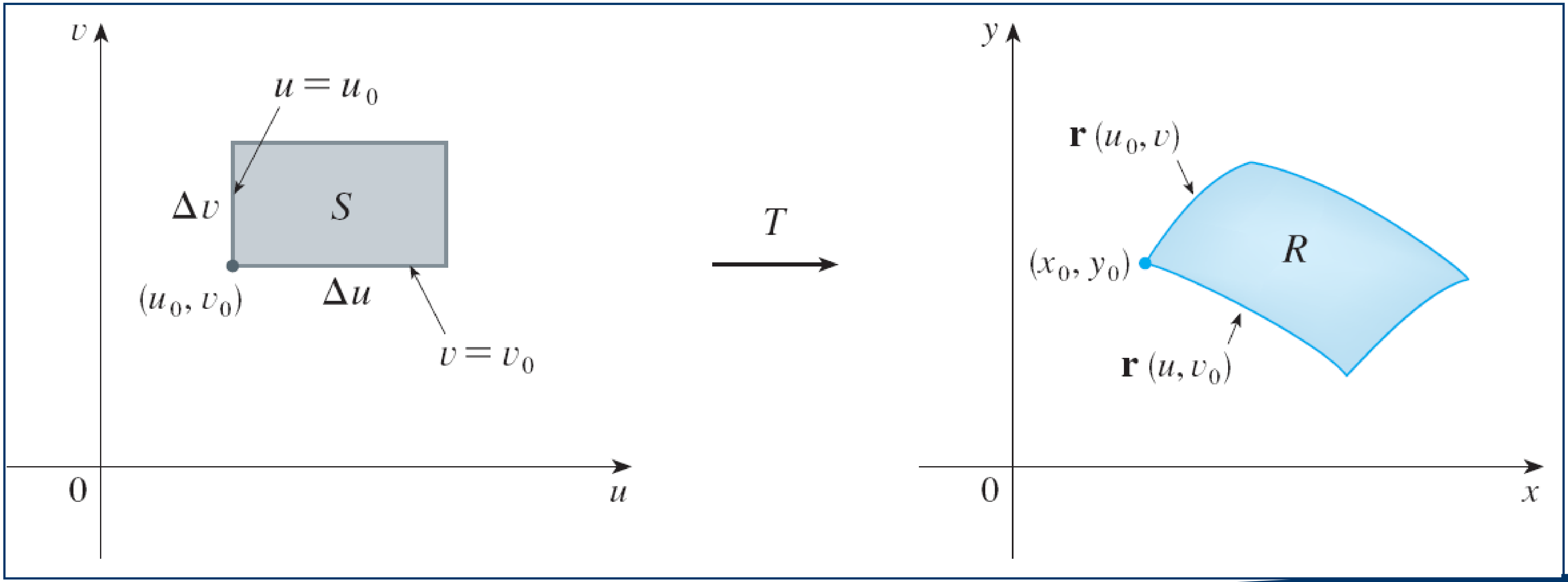


FIGURE 3

- We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)$$

$$\mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

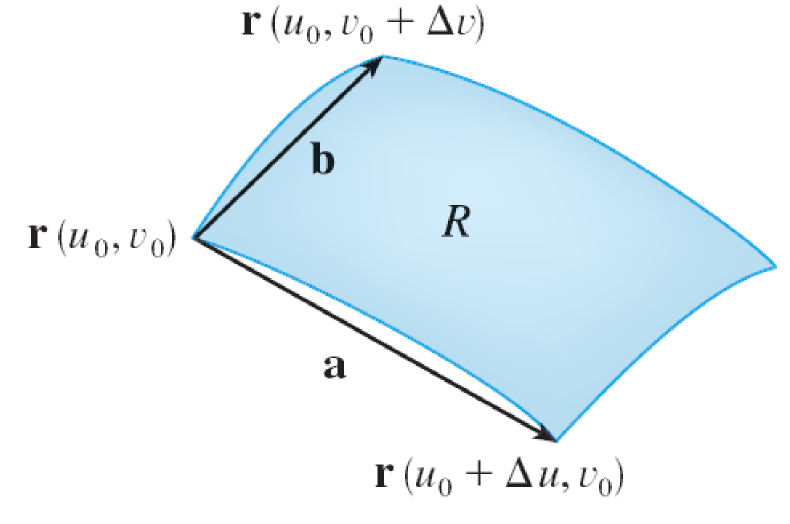


FIGURE 4

DOUBLE INTEGRALS

- However,

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

- So,

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \, \mathbf{r}_u$$

  - Similarly,

$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \, \mathbf{r}_v$$

DOUBLE INTEGRALS

- This means that we can approximate *R* by a parallelogram determined by the vectors

$$\Delta u \, \mathbf{r}_u \text{ and } \Delta v \, \mathbf{r}_v$$

- See Figure 5.

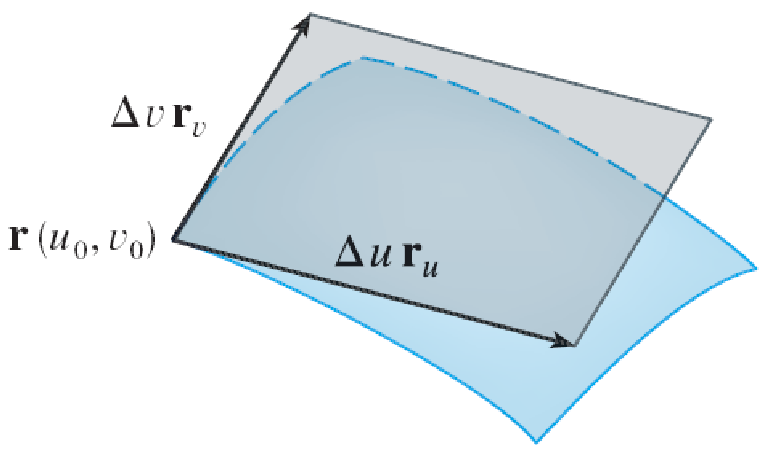


FIGURE 5

- Thus, we can approximate the area of *R* by the area of this parallelogram, which, from Section 10.4, is

$$|(\Delta u\, \mathbf{r}_u) \times (\Delta v\, \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$$

DOUBLE INTEGRALS

❖Computing the cross product, we obtain:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial u} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial u} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

JACOBIAN

- The determinant that arises in this calculation is called the *Jacobian* of the transformation.
  - It is given a special notation.

Definition 7

The **Jacobian** of the transformation T given by

$x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

JACOBIAN OF *T*

- With this notation, we can use Equation 6 to give an approximation to the area ΔA of R:

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

JACOBIAN

- The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851).
  - The French mathematician Cauchy first used these special determinants involving partial derivatives.
  - Jacobi, though, developed them into a method for evaluating multiple integrals.

DOUBLE INTEGRALS

❖ Next, we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}.

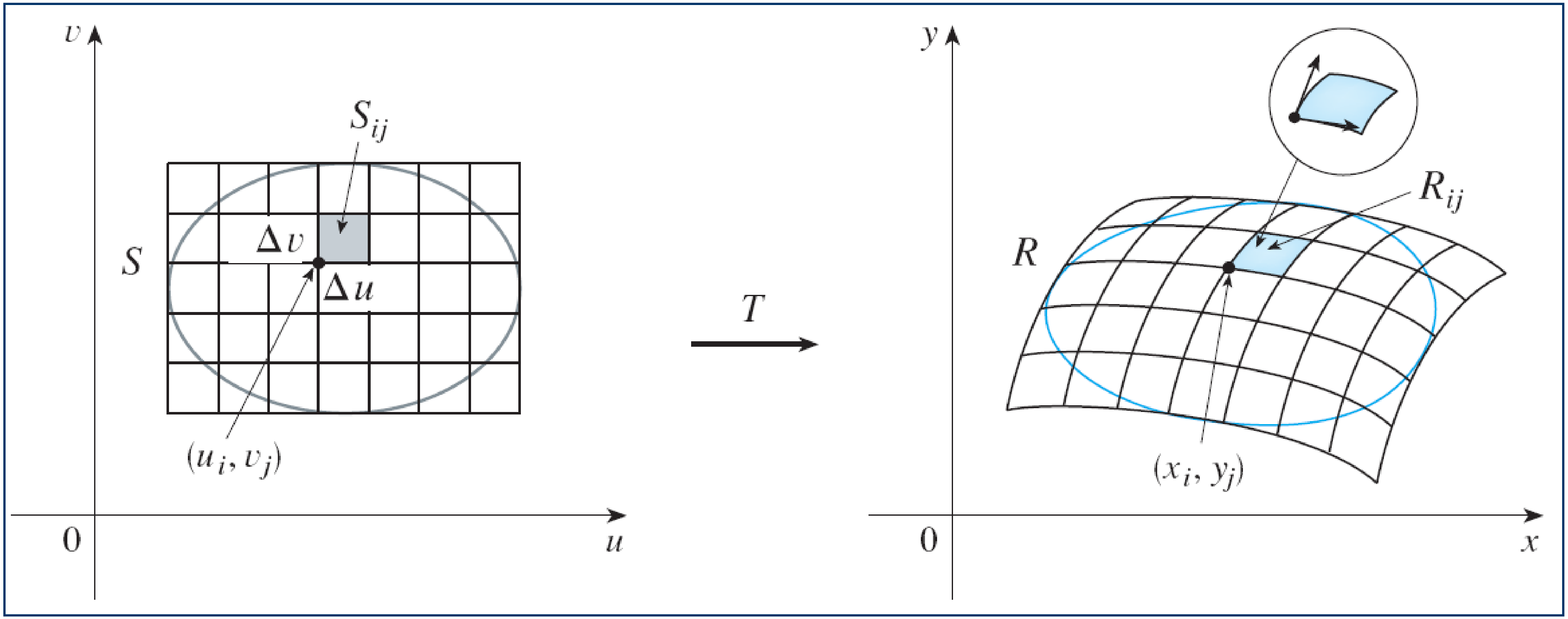


FIGURE 6

DOUBLE INTEGRALS

❖Applying Approximation 8 to each R_{ij} , we approximate the double integral of f over R as follows.

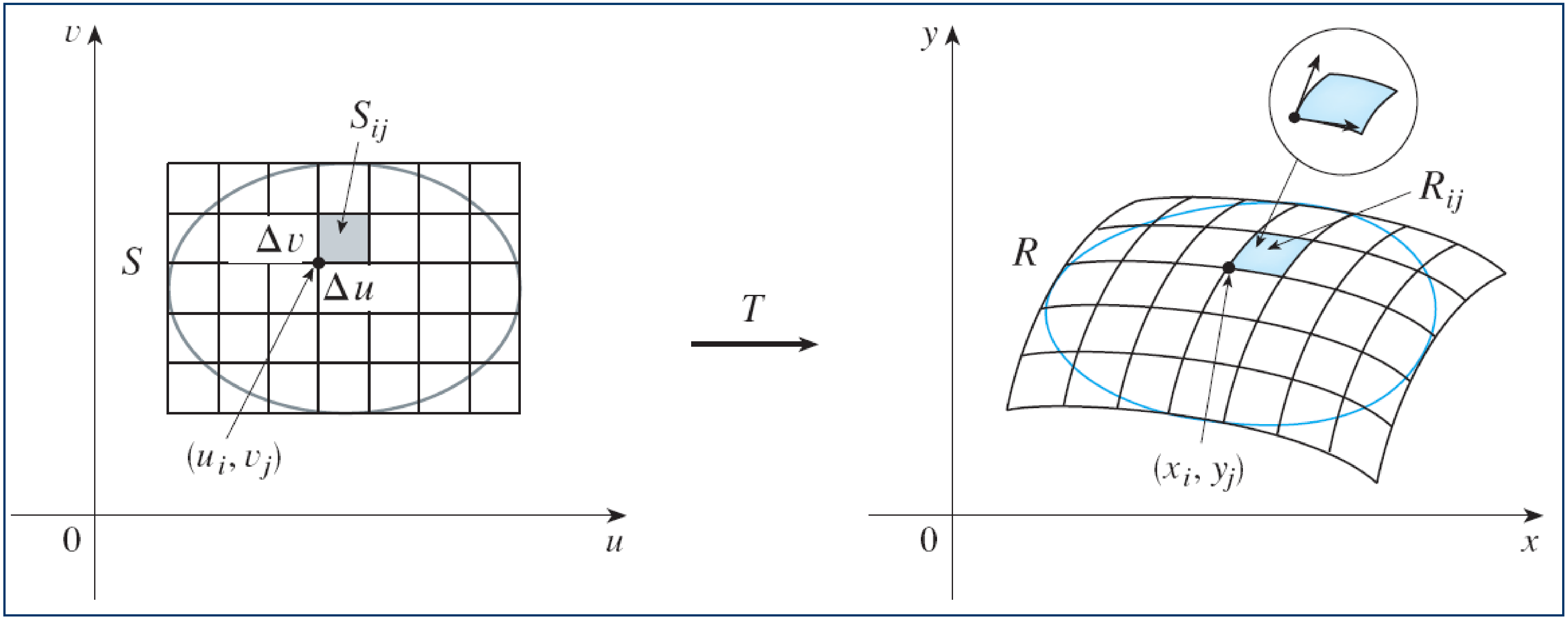


FIGURE 6

DOUBLE INTEGRALS

$$\iint_R f(x, y)\, dA$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

- where the Jacobian is evaluated at (u_i, v_j).

DOUBLE INTEGRALS

- Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u,v),h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

- The foregoing argument suggests that the following theorem is true.
  - A full proof is given in books on advanced calculus.

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

- Suppose T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose f is continuous on R and that R and S are type I or type II plane regions. Suppose T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

- Theorem 9 says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing:

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

- Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2.
  - Instead of the derivative dx/du, we have the absolute value of the Jacobian, that is,

$$|\partial(x, y)/\partial(u, v)|$$

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

- As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case.

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

- Here, the transformation T from the $r\theta$-plane to the xy-plane is given by:

$$x = g(r, \theta) = r \cos \theta$$
$$y = h(r, \theta) = r \sin \theta$$

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

- The geometry of the transformation is shown in Figure 7.
  - T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane.

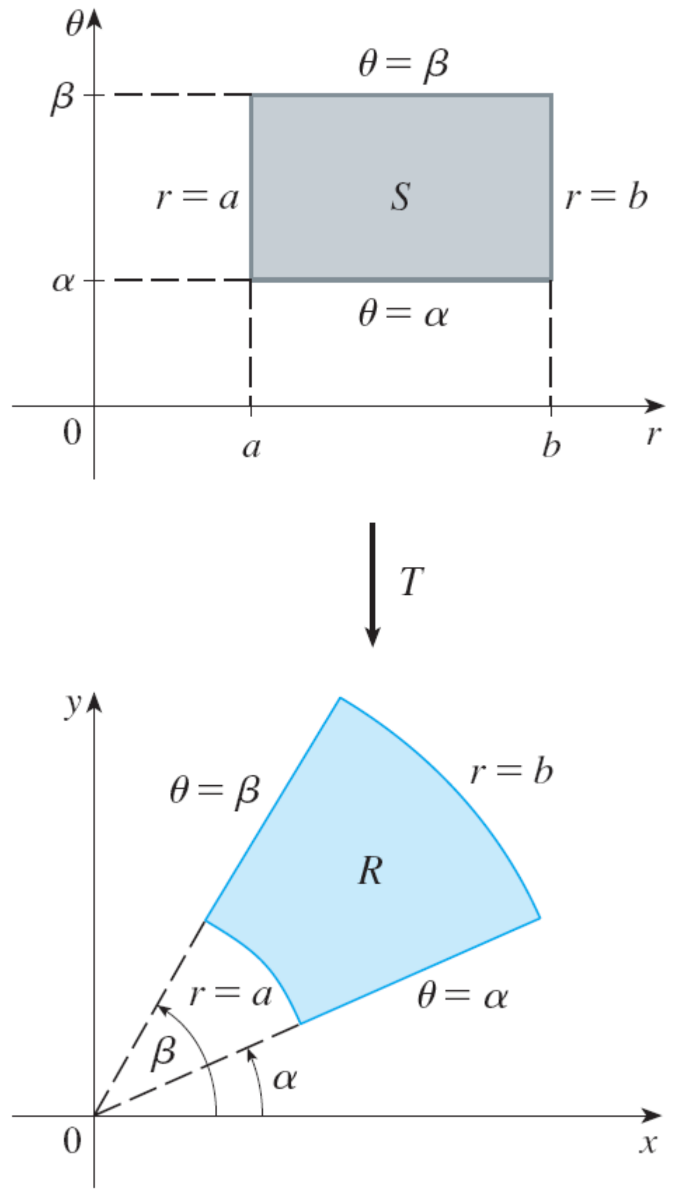


FIGURE 7
The polar coordinate transformation

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

❖The Jacobian of *T* is:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial x}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r > 0$$

CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

- So, Theorem 9 gives:

$$\iint_R f(x,y)\,dx\,dy$$

$$= \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dr\,d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

  - This is the same as Formula 2 in Section 12.3

Example 2

- Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$ where R is the region bounded by:
  - The x-axis.
  - The parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

Example 2 SOLUTION

- The region R is pictured in Figure 2.

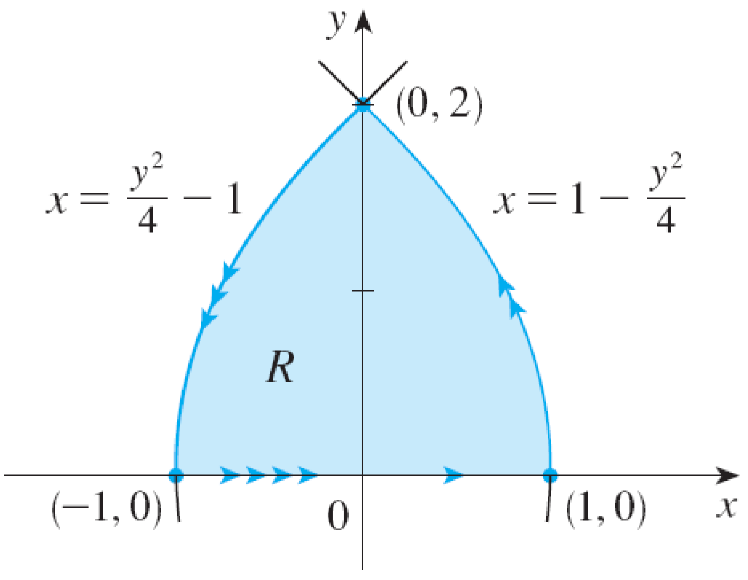


FIGURE 2

Example 2 SOLUTION

- In Example 1, we discovered that

$$T(S) = R$$

where S is the square $[0, 1] \times [0, 1]$.

  - Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region than R.

Example 2 SOLUTION

- First, we need to compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4u^2 + 4v^2 > 0$$

Example 2 SOLUTION

❖ So, by Theorem 9,

$$\iint_R y\,dA = \iint_S 2uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

$$= \int_0^1 \int_0^1 (2uv)4(u^2+v^2)\,du\,dv$$

$$= 8\int_0^1 \int_0^1 (u^3v + uv^3)\,du\,dv$$

$$= 8\int_0^1 \left[\tfrac{1}{4}u^4v + \tfrac{1}{2}u^2v^3\right]_{u=0}^{u=1} dv$$

$$= \int_0^1 (2v+4v^3)\,dv = \left[v^2+v^4\right]_0^1 = 2$$

Note

- Example 2 was not very difficult to solve as we were given a suitable change of variables.
- If we are not supplied with a transformation, the first step is to think of an appropriate change of variables.

Note

- If $f(x, y)$ is difficult to integrate,
  - The form of $f(x, y)$ may suggest a transformation.
- If the region of integration R is awkward,
  - The transformation should be chosen so that the corresponding region S in the uv-plane has a convenient description.

Example 3

❖ Evaluate the integral

$$\iint_R e^{(x+y)/(x-y)} dA$$

where R is the trapezoidal region with vertices (1, 0), (2, 0), (0, –2), (0,–1)

Example 3 SOLUTION

- It isn't easy to integrate $e^{(x+y)/(x-y)}$.
- So, we make a change of variables suggested by the form of this function:

$$u = x + y \qquad v = x - y$$

  - These equations define a transformation T^{-1} from the xy-plane to the uv-plane.

Example 3 SOLUTION

- Theorem 9 talks about a transformation *T* from the *uv*-plane to the *xy*-plane.
- It is obtained by solving Equations 10 for *x* and *y*:

$$x = \tfrac{1}{2}(u + v) \qquad y = \tfrac{1}{2}(u - v)$$

Example 3 SOLUTION

- The Jacobian of T is:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Example 3 SOLUTION

- To find the region *S* in the *uv*-plane corresponding to *R*, we note that:
  - The sides of *R* lie on the lines

    $$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

  - From either Equations 10 or Equations 11, the image lines in the *uv*-plane are:

    $$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Example 3 SOLUTION

- Thus, the region S is the trapezoidal region with vertices (1, 1), (2, 2), (–2, 2), (–1 ,1) shown in Figure 8.
- $S = \{(u, v) \mid 1 \le v \le 2, \; -v \le u \le v\}$

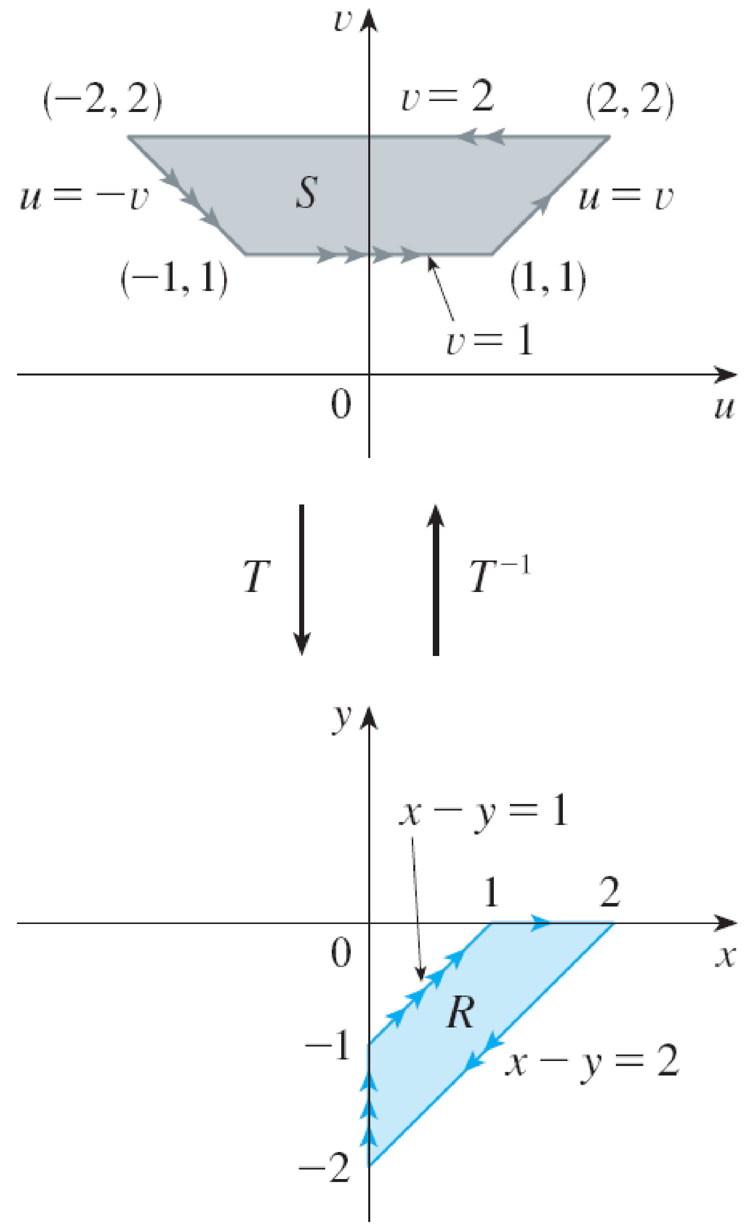


FIGURE 8

Example 3 SOLUTION

- So, Theorem 9 gives:

$$\iint_R e^{(x+y)/(x-y)}\,dA = \iint_S e^{u/v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v}\left(\tfrac{1}{2}\right) du\,dv$$

$$= \tfrac{1}{2}\int_1^2 \left[ve^{u/v}\right]_{u=-v}^{u=v} dv$$

$$= \tfrac{1}{2}\int_1^2 (e-e^{-1})v\,dv = \tfrac{3}{4}(e-e^{-1})$$

TRIPLE INTEGRALS

- There is a similar change of variables formula for triple integrals.
  - Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations
    $x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$

TRIPLE INTEGRALS

- The **Jacobian** of T is this 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Formula 13

- Under hypotheses similar to those in Theorem 9, we have this formula for triple integrals:

$$\iiint_R f(x, y, z)\, dV$$

$$= \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Example 4

- Use Formula 13 to derive the formula for triple integration in spherical coordinates.

- SOLUTION
  - The change of variables is given by:

$$x = \rho \sin \phi \cos \theta$$
$$y = \rho \sin \phi \sin \theta$$
$$z = \rho \cos \phi$$

Example 4 SOLUTION

❖ We compute the Jacobian as follows:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$$

$$=\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

Example 4 SOLUTION

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix}$$

$$-\rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta)$$

$$-\rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi$$

$$= -\rho^2\sin\phi$$

Example 4 SOLUTION

- Since $0 \le \phi \le \pi$, we have $\sin \phi \ge 0$.
- Therefore,

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \left| -\rho^2 \sin \phi \right| = \rho^2 \sin \phi$$

Example 4 SOLUTION

- Thus, Formula 13 gives:

$$\iiint_R f(x, y, z)\, dV$$

$$= \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \; d\theta \; d\phi$$

  - This is equivalent to Formula 3 in Section12.7.