14

PARTIAL DERIVATIVES

One of the most important ideas in single-variable calculus is:

- As we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line.

- We can then approximate the function by a linear function.

Here, we develop similar ideas in three dimensions.

- As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane).

- We can then approximate the function by a linear function of two variables.

We also extend the idea of a differential to functions of two or more variables.

14.4

Tangent Planes and Linear Approximations

In this section, we will learn how to:

Approximate functions using tangent planes and linear functions.

TANGENT PLANES

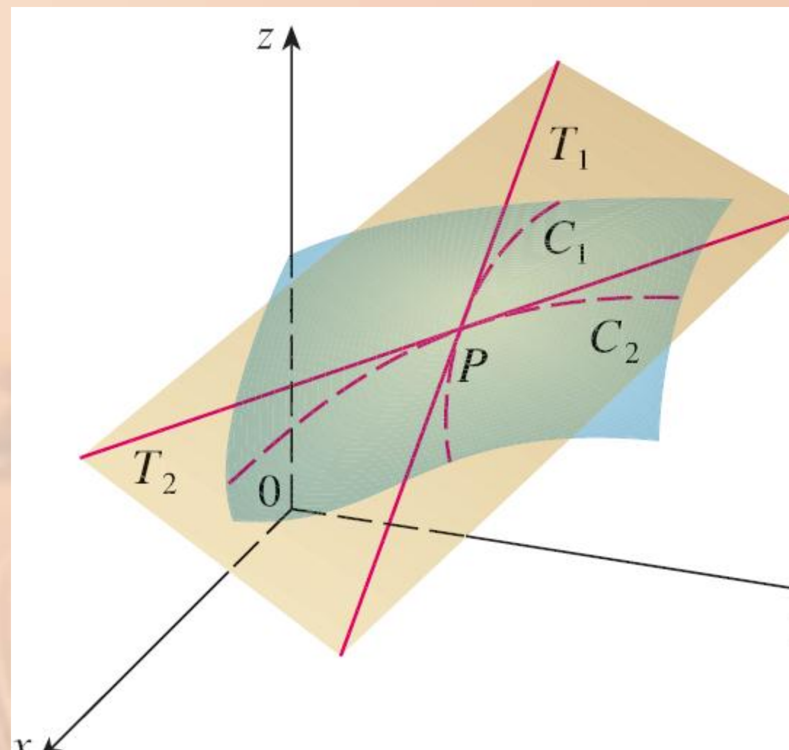
Suppose a surface *S* has equation $z = f(x, y)$, where *f* has continuous first partial derivatives.

Let $P(x_0, y_0, z_0)$ be a point on *S*.

TANGENT PLANES

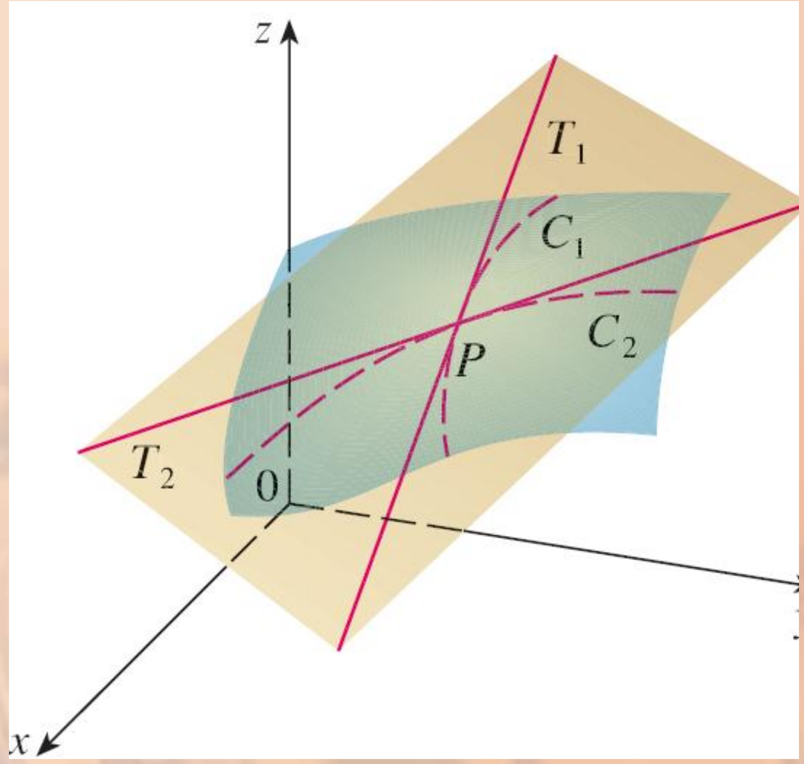
As in Section 10.3, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S.

- Then, the point P lies on both C_1 and C_2.



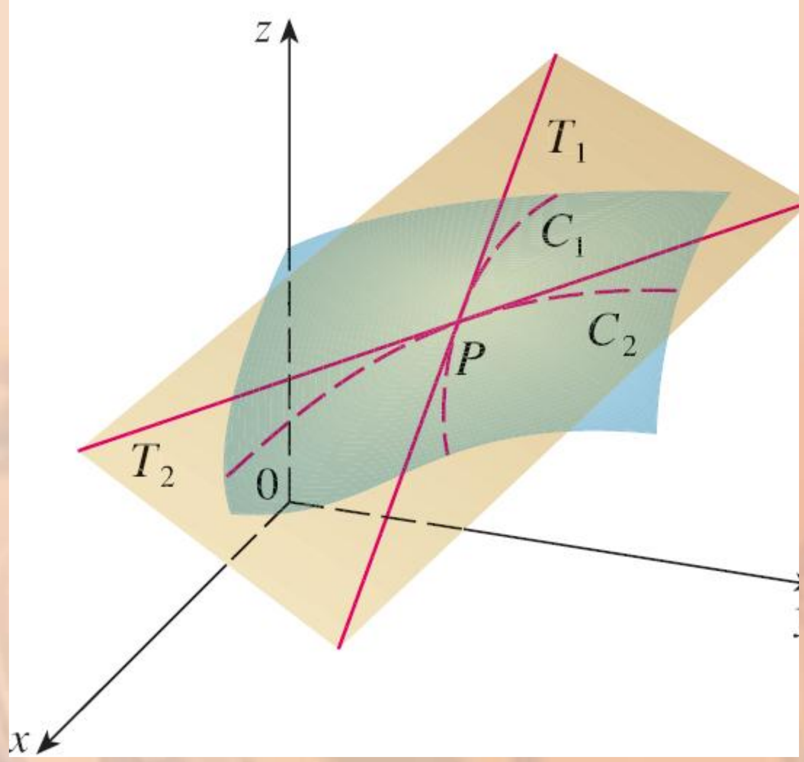
TANGENT PLANES

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.



TANGENT PLANE

Then, the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.



TANGENT PLANES

We will see in Section 10.6 that, if *C* is any other curve that lies on the surface *S* and passes through *P*, then its tangent line at *P* also lies in the tangent plane.

TANGENT PLANES

Therefore, you can think of the tangent plane to *S* at *P* as consisting of all possible tangent lines at *P* to curves that lie on *S* and pass through *P*.

- The tangent plane at *P* is the plane that most closely approximates the surface *S* near the point *P*.

TANGENT PLANES

We know from Equation 7 in Section 12.5 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

By dividing that equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

TANGENT PLANES

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1.

Setting $y = y_0$ in Equation 1 gives:

$$z - z_0 = a(x - x_0)$$
$$y = y_0$$

- We recognize these as the equations (in point-slope form) of a line with slope *a*.

However, from Section 10.3, we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$.

- Therefore, $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get:

$$z - z_0 = b(y - y_0)$$

This must represent the tangent line T_2.

- Thus, $b = f_y(x_0, y_0)$.

TANGENT PLANES

Equation 2

Suppose *f* has continuous partial derivatives.

An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

- Let $f(x, y) = 2x^2 + y^2$.
- Then,

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

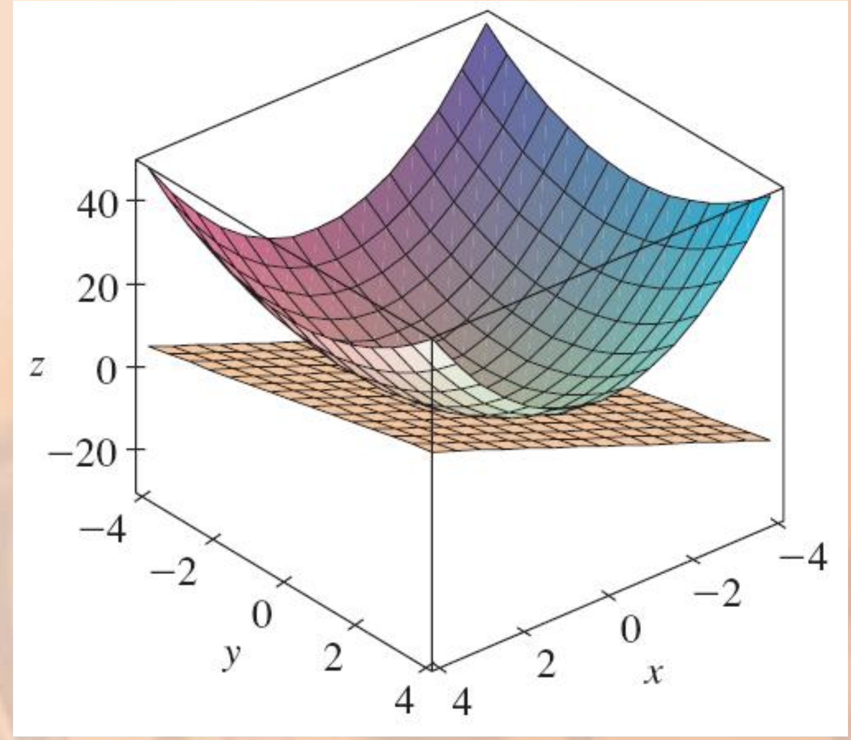
- So, Equation 2 gives the equation of the tangent plane at (1, 1, 3) as:

$$z - 3 = 4(x - 1) + 2(y - 1)$$

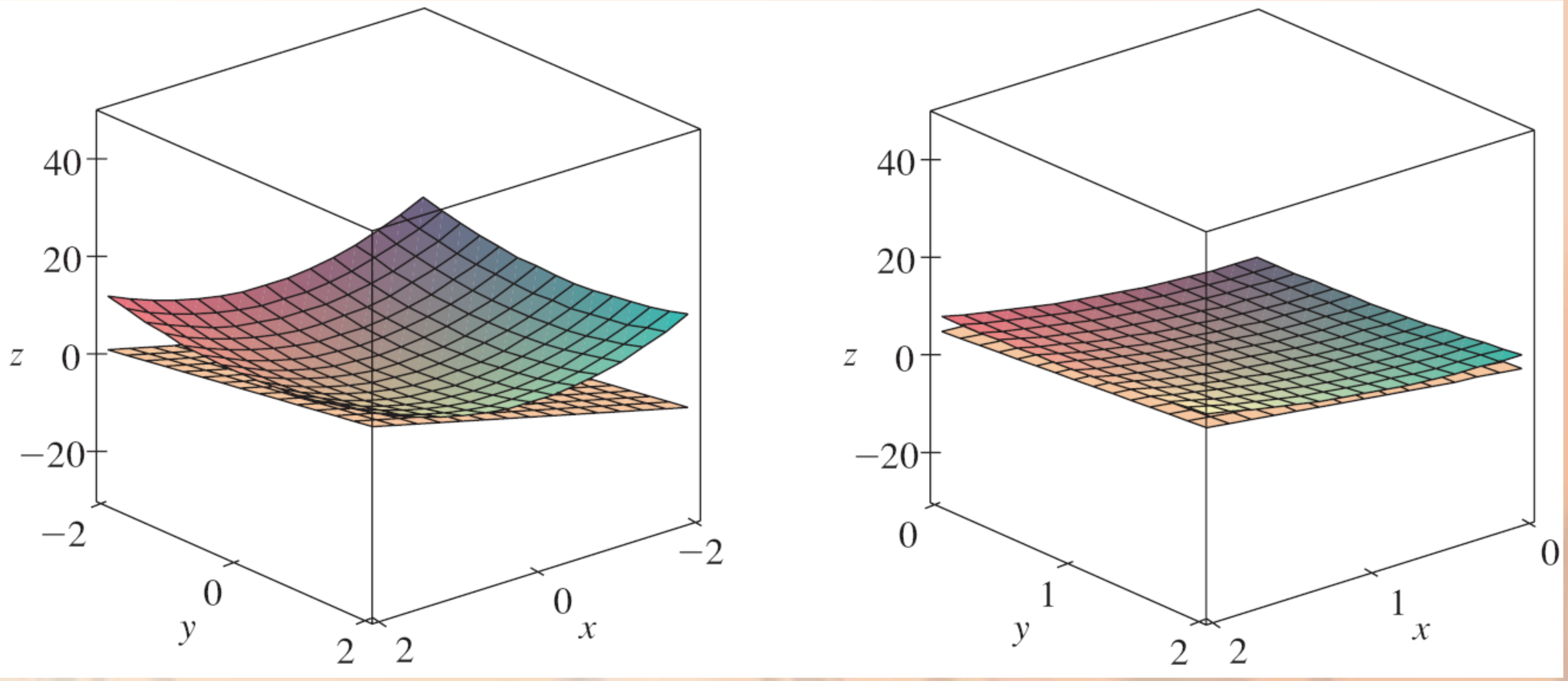
or

$$z = 4x + 2y - 3$$

The figure shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1.

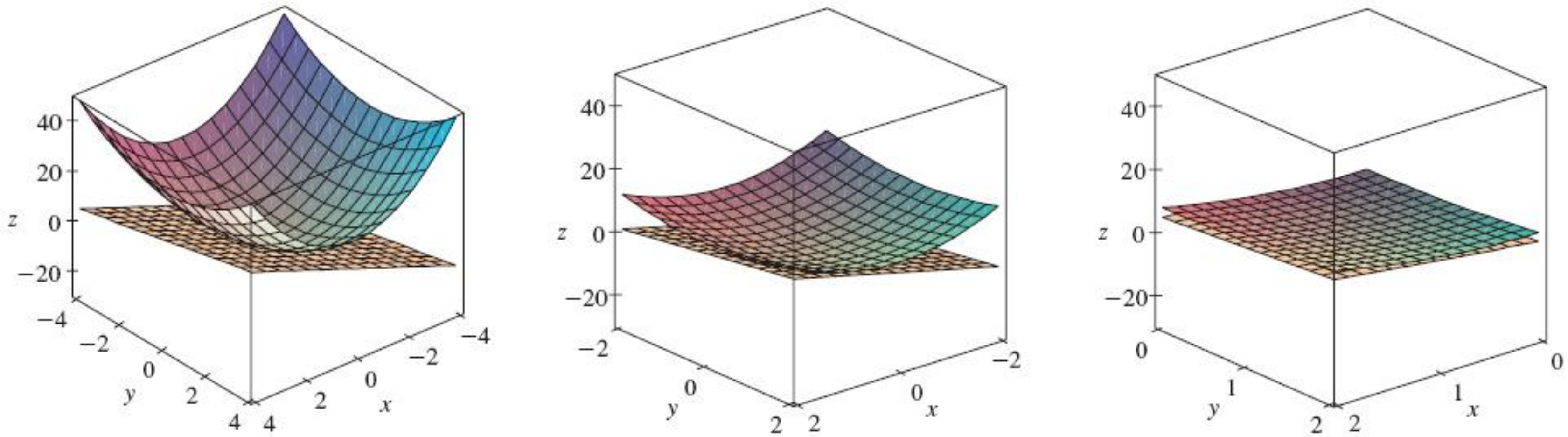


Here, we zoom in toward the point by restricting the domain of the function $f(x, y) = 2x^2 + y^2$.

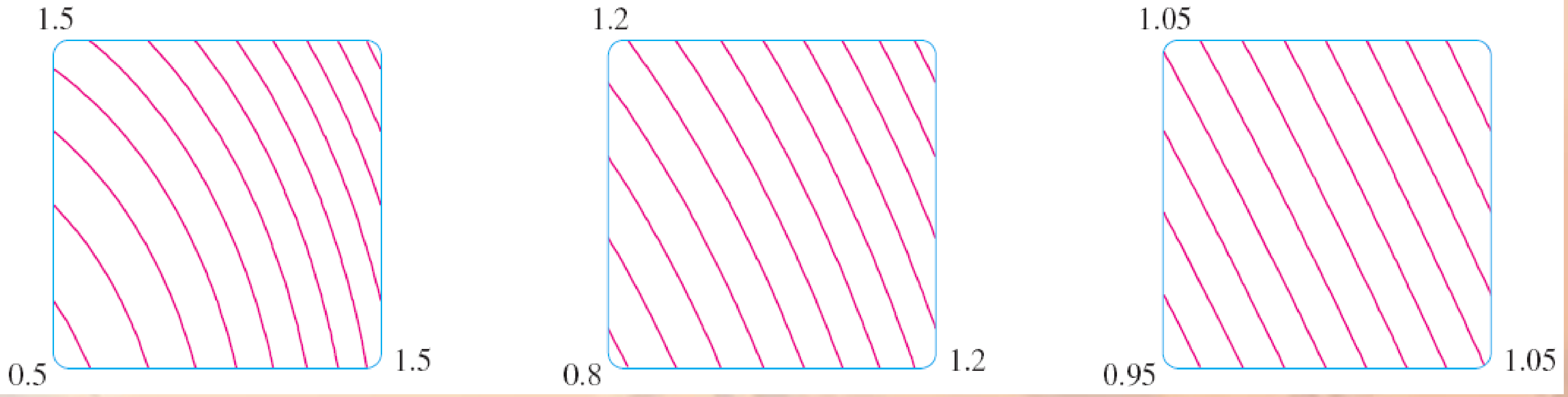


Notice that, the more we zoom in,

- The flatter the graph appears.
- The more it resembles its tangent plane.

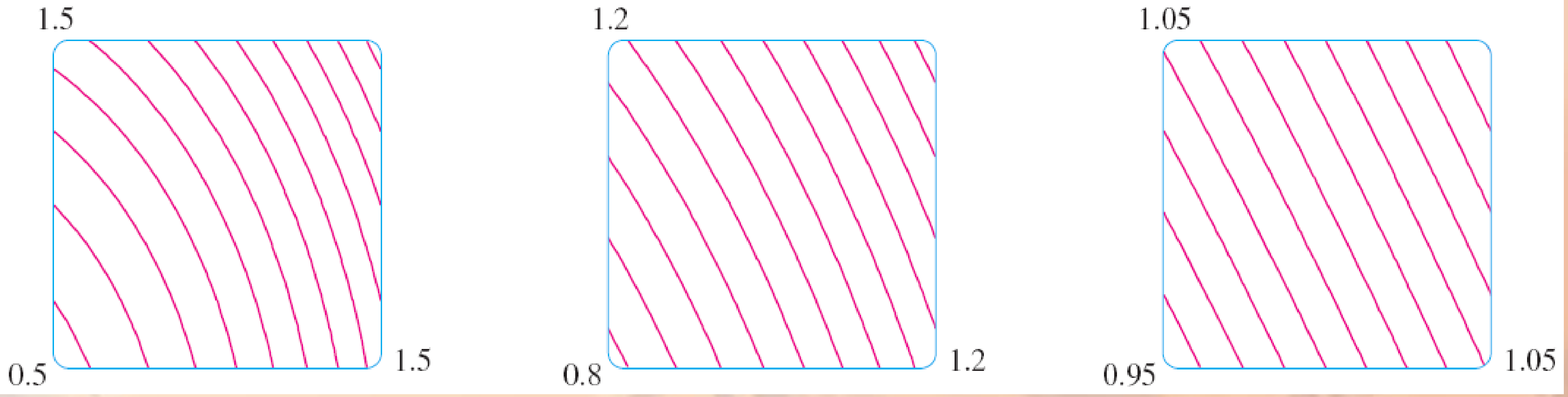


Here, we corroborate that impression by zooming in toward the point (1, 1) on a contour map of the function $f(x, y) = 2x^2 + y^2$.



TANGENT PLANES

Notice that, the more we zoom in, the more the level curves look like equally spaced parallel lines—characteristic of a plane.



In Example 1, we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is:

$$z = 4x + 2y - 3$$

Thus, in view of the visual evidence in the previous two figures, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near $(1, 1)$.

LINEARIZATION & LINEAR APPROXIMATION

The function *L* is called the linearization of *f* at (1, 1).

The approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the linear approximation or tangent plane approximation of *f* at (1, 1).