6

APPLICATIONS OF INTEGRATION

6.5
Average Value of a Function

In this section, we will learn about:
Applying integration to find out the average value of a function.

AVERAGE VALUE OF A FUNCTION

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{ave} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

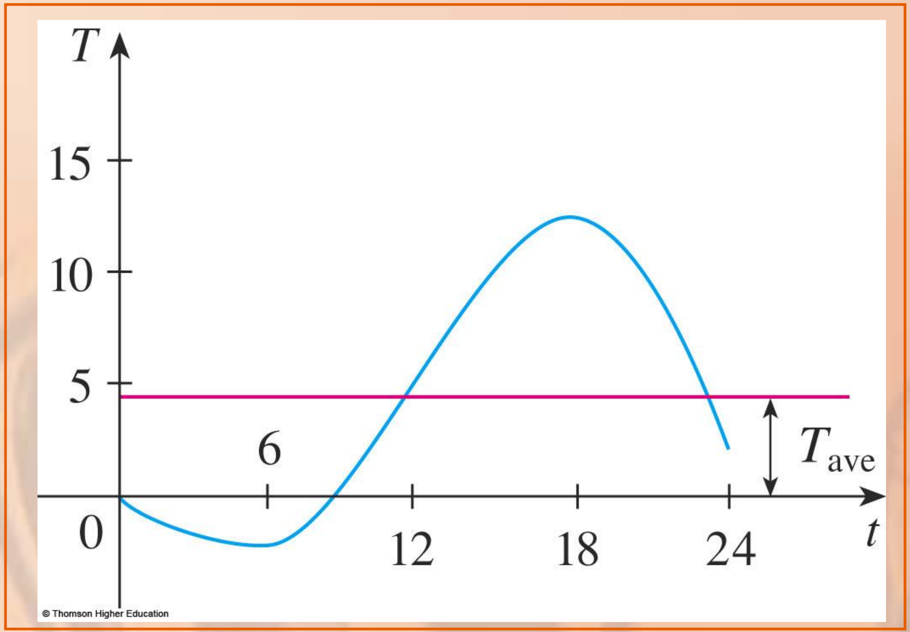
AVERAGE VALUE OF A FUNCTION

However, how do we compute the average temperature during a day if infinitely many temperature readings are possible?

AVERAGE VALUE OF A FUNCTION

This figure shows the graph of a temperature function $T(t)$, where:

- t is measured in hours
- T in °C
- T_{ave}, a guess at the average temperature



AVERAGE VALUE OF A FUNCTION

In general, let's try to compute the average value of a function $y = f(x)$, $a \le x \le b$.

AVERAGE VALUE OF A FUNCTION

We start by dividing the interval [*a*, *b*] into *n* equal subintervals, each with length $\Delta x = (b - a) / n$.

AVERAGE VALUE OF A FUNCTION

Then, we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and calculate the average of the numbers $f(x_i^*), \ldots, f(x_n^*)$:

$$\frac{f(x_i^*) + \cdots + f(x_n^*)}{n}$$

- For example, if *f* represents a temperature function and $n = 24$, then we take temperature readings every hour and average them.

AVERAGE VALUE OF A FUNCTION

Since $\Delta x = (b - a) / n$, we can write $n = (b - a) / \Delta x$ and the average value becomes:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}}$$

$$= \frac{1}{b-a}\left[f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

AVERAGE VALUE OF A FUNCTION

If we let *n* increase, we would be computing the average value of a large number of closely spaced values.

- For example, we would be averaging temperature readings taken every minute or even every second.

By the definition of a definite integral, the limiting value is:

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

AVERAGE VALUE OF A FUNCTION

So, we define the average value of *f* on the interval [*a*, *b*] as:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 1

Find the average value of the function $f(x) = 1 + x^2$ on the interval [-1, 2].

With a = -1 and b = 2,

we have:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2}\left(1+x^2\right)\,dx$$

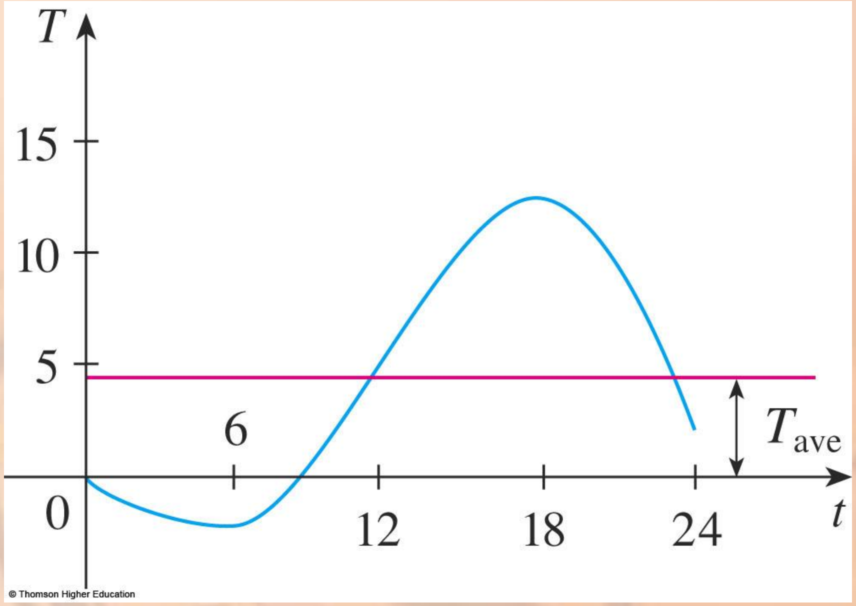
$$= \frac{1}{3}\left[x+\frac{x^3}{3}\right]_{-1}^{2} = 2$$

AVERAGE VALUE

If $T(t)$ is the temperature at time t, we might wonder if there is a specific time when the temperature is the same as the average temperature.

AVERAGE VALUE

For the temperature function graphed here, we see that there are two such times—just before noon and just before midnight.



AVERAGE VALUE

In general, is there a number c at which the value of a function f is exactly equal to the average value of the function—that is, $f(c) = f_{ave}$?

The mean value theorem for integrals states that this is true for continuous functions.

MEAN VALUE THEOREM

If *f* is continuous on [*a*, *b*], then there exists a number *c* in [*a*, *b*] such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

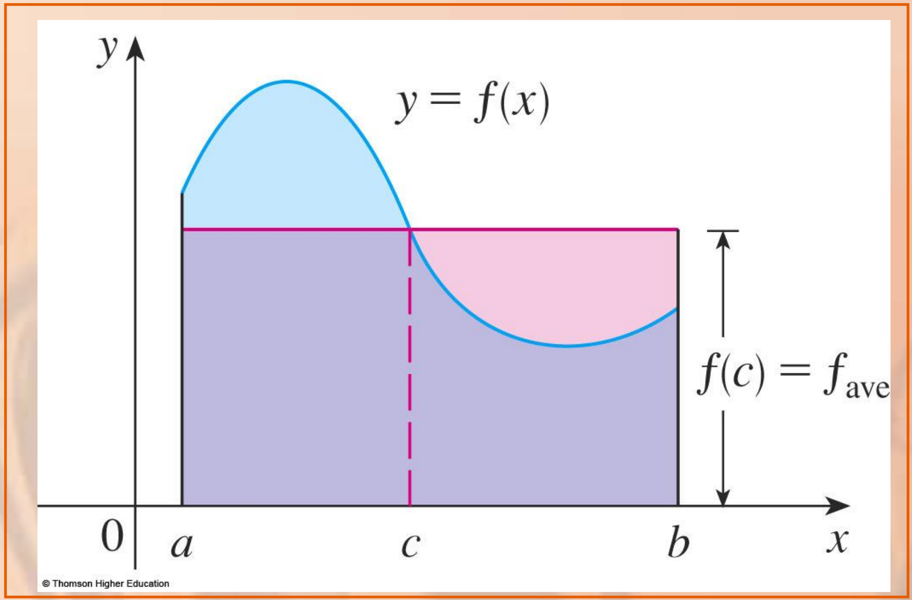
MEAN VALUE THEOREM

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for derivatives and the Fundamental Theorem of Calculus.

MEAN VALUE THEOREM

The geometric interpretation of the Mean Value Theorem for Integrals is as follows.

- For ‘positive’ functions *f*, there is a number *c* such that the rectangle with base [*a*, *b*] and height *f*(*c*) has the same area as the region under the graph of *f* from *a* to *b*.



Example 2

Since $f(x) = 1 + x^2$ is continuous on the interval [-1, 2], the Mean Value Theorem for Integrals states there is a number *c* in [-1, 2] such that:

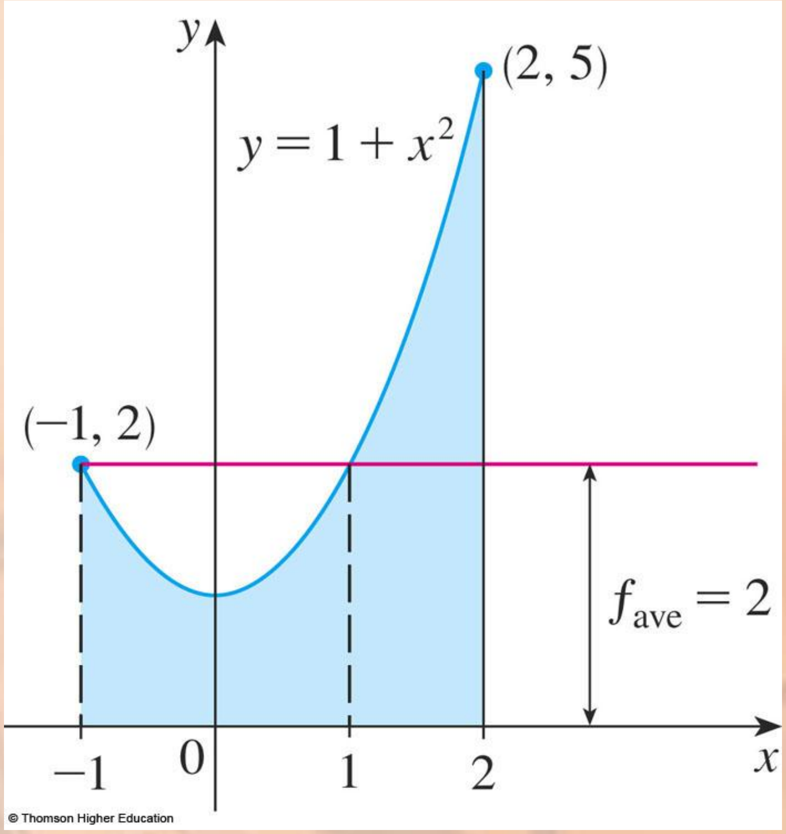
$$\int_{-1}^{2} \left(1 + x^2\right) dx = f(c)[2 - (-1)]$$

In this particular case, we can find *c* explicitly.

- From Example 1, we know that $f_{ave} = 2$.
- So, the value of *c* satisfies $f(c) = f_{ave} = 2$.
- Therefore, $1 + c^2 = 2$.
- Thus, $c^2 = 1$.

So, in this case, there happen to be two numbers $c = \pm 1$ in the interval [-1, 2] that work in the Mean Value Theorem for Integrals.

Examples 1 and 2 are illustrated here.



Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

If $s(t)$ is the displacement of the car at time t, then by definition, the average velocity of the car over the interval is:

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

MEAN VALUE THEOREM

Example 3

On the other hand, the average value of the velocity function on the interval is:

$$v_{ave} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\ dt = \frac{1}{t_2 - t_1}\int_{t^1}^{t_2} s'(t)\ dt$$

$$= \frac{1}{t_2 - t_1}\left[s(t_2) - s(t_1)\right] \qquad \left(\text{by the Net Change Theorem}\right)$$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \text{average velocity}$$