CHAPTER 13 VECTOR CALCULUS

LINE INTEGRALS

LINE INTEGRALS

LINE INTEGRALS

- We start with a plane curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

- Equivalently, C can be given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$.
- We assume that C is a smooth curve.
  - This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.
  - See Section 10.7

LINE INTEGRALS

- Let's divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width.
- We let $x_i = x(t_i)$ and $y_i = y(t_i)$.

LINE INTEGRALS

- Then, the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$.
- See Figure 1.

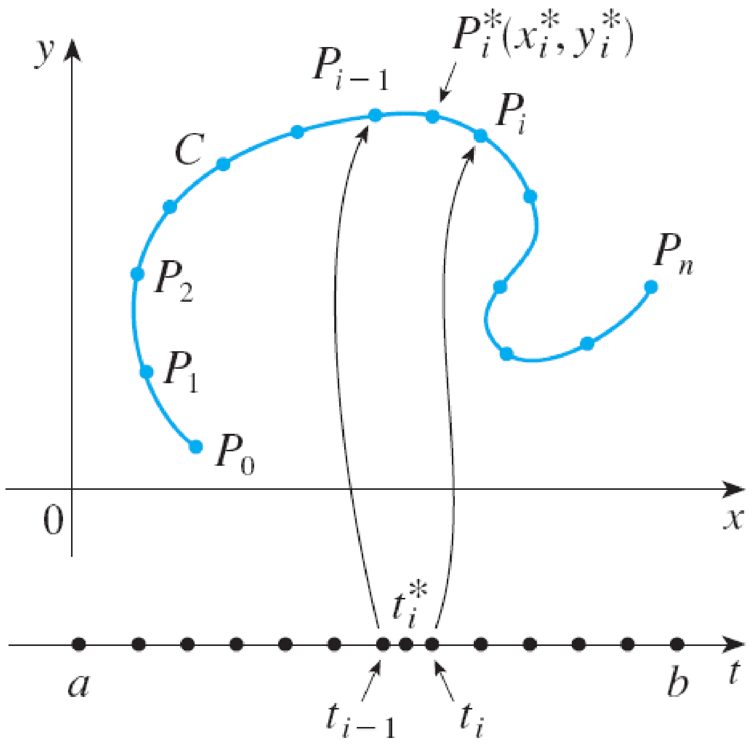

FIGURE 1

- We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc.
  - This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.

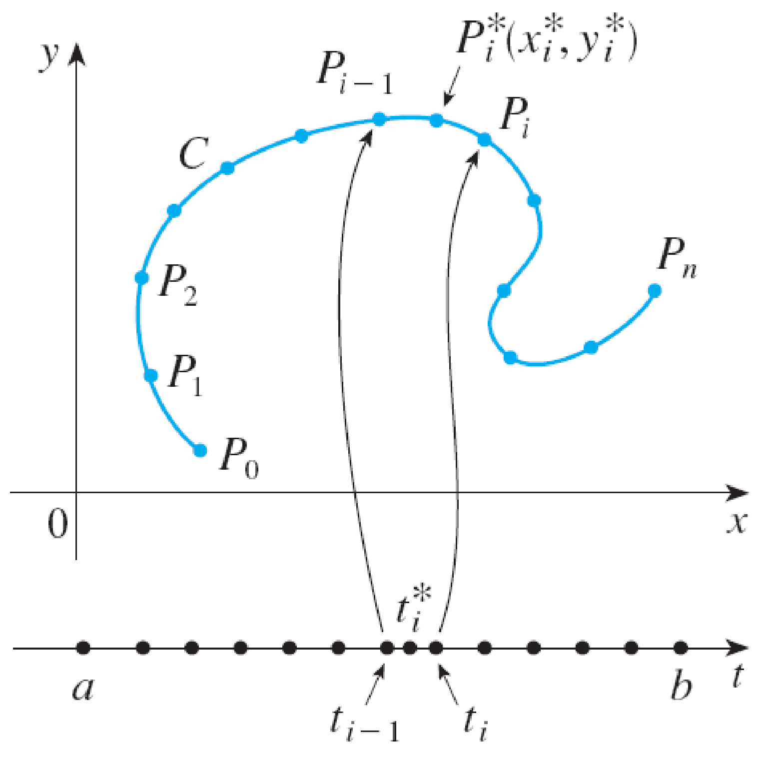

FIGURE 1

LINE INTEGRALS

- Now, if f is any function of two variables whose domain includes the curve C, we:
  1. Evaluate f at the point (x_i^*, y_i^*).
  2. Multiply by the length Δs_i of the subarc.
  3. Form the sum $\sum_{i=1}^{n} f\left(x_i^*, y_i^*\right)\Delta s_i$

     which is similar to a Riemann sum.
- Then, we take the limit of these sums and make the following definition by analogy with a single integral.

Definition 2

If *f* is defined on a smooth curve *C* given by Equations 1, the **line integral of *f* along *C*** is:

$$\int_C f(x,y)\,ds = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} f\left(x_i^*, y_i^*\right)\Delta s_i$$

if this limit exists.

LINE INTEGRALS

❖In Section 9.2, we found that the length of *C* is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- A similar type of argument can be used to show that, if *f* is a continuous function, then the limit in Definition 2 always exists.

Formula 3

❖ Then, this formula can be used to evaluate the line integral.

$$\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

LINE INTEGRALS

- The value of the line integral does not depend on the parametrization of the curve—provided the curve is traversed exactly once as t increases from a to b.
- If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

LINE INTEGRALS

- So, the way to remember Formula 3 is to express everything in terms of the parameter t :
  - Use the parametric equations to express x and y in terms of t and write ds as:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

LINE INTEGRALS

- In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as:

$$x = x$$
$$y = 0$$
$$a \leq x \leq b$$

LINE INTEGRALS

- Formula 3 then becomes

$$\int_C f(x,y)\,ds = \int_a^b f(x,0)\,dx$$

  - So, the line integral reduces to an ordinary single integral in this case.

- Just as for an ordinary single integral, we can interpret the line integral of a *positive function* as an area.

LINE INTEGRALS

- In fact, if $f(x, y) \geq 0$, $\int_C f(x, y)\,ds$ represents the area of one side of the "fence" or "curtain" shown in Figure 2, whose:
  - Base is C.
  - Height above the point (x, y) is $f(x, y)$.

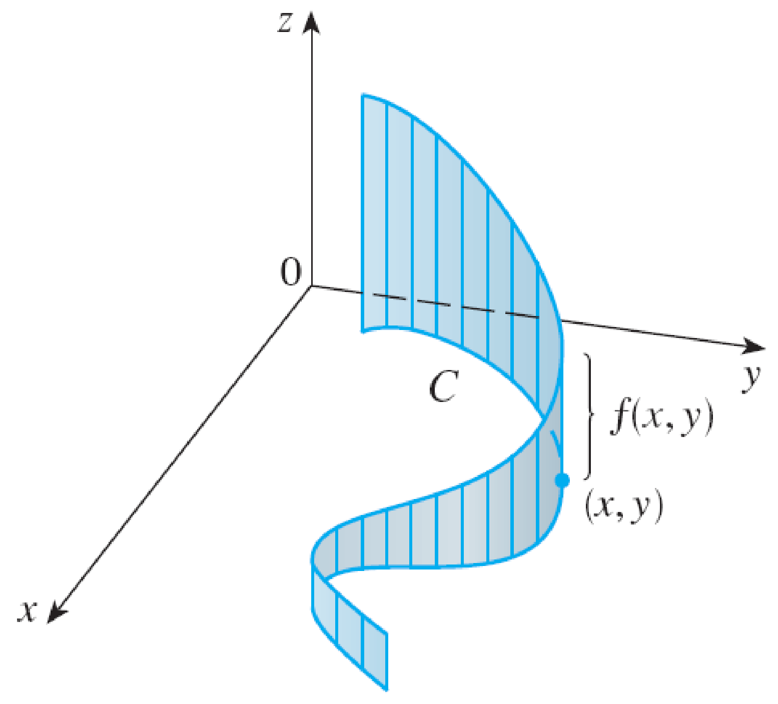

FIGURE 2

Example 1

- Evaluate

$$\int_C \left(2 + x^2 y\right) ds$$

where C is the upper half of the unit circle $x^2 + y^2 = 1$

- SOLUTION
  - To use Formula 3, we first need parametric equations to represent C.

Example 1 SOLUTION

- Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \qquad y = \sin t$$

- Also, the upper half of the circle is described by the parameter interval

$$0 \leq t \leq \pi$$

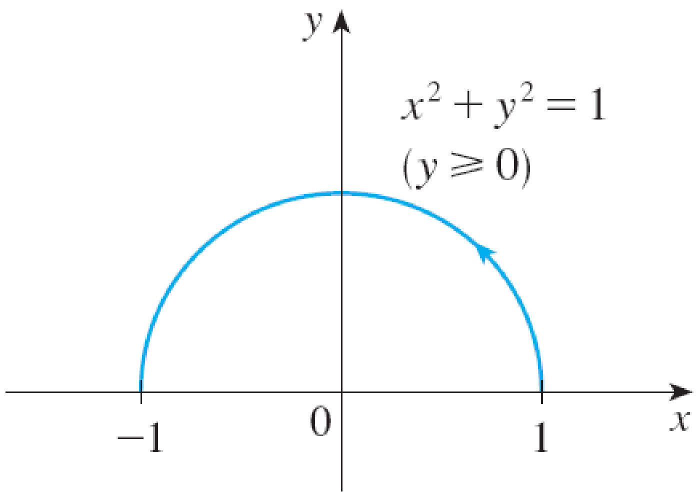

FIGURE 3

Example 1 SOLUTION

- So, Formula 3 gives:

$$\int_C \left(2 + x^2 y\right) ds = \int_0^{\pi} \left(2 + \cos^2 t \sin t\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \left(2 + \cos^2 t \sin t\right) \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{\pi} \left(2 + \cos^2 t \sin t\right) dt$$

$$= \left[2t - \frac{\cos^3 t}{3}\right]_0^{\pi} = 2\pi + \tfrac{2}{3}$$

PIECEWISE-SMOOTH CURVE

- Now, let C be a **piecewise-smooth curve**.
  - That is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i.

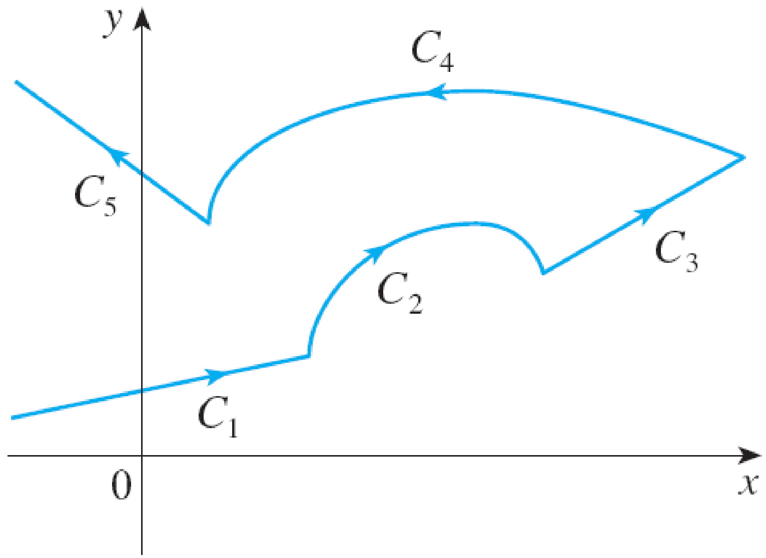

FIGURE 4

A piecewise-smooth curve

LINE INTEGRALS

❖ Then, we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,ds$$

$$= \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \dots + \int_{C_n} f(x,y)\,ds$$

Example 2

- Evaluate

$$\int_C 2x\,ds$$

where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

Example 2 SOLUTION

- The curve is shown in Figure 5.
- C_1 is the graph of a function of *x*.
  - So, we can choose *x* as the parameter.
  - Then, the equations for C_1 become:

$$x = x \qquad y = x^2 \qquad 0 \leq x \leq 1$$

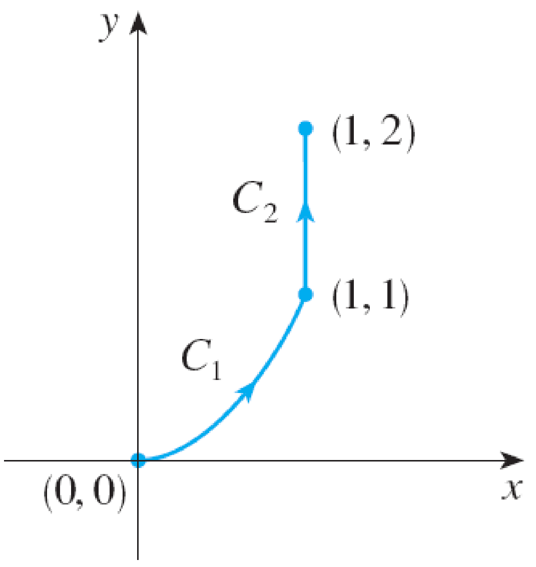

FIGURE 5
$C = C_1 \cup C_2$

Example 2 SOLUTION

- Therefore,

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 2x\sqrt{1+4x^2}\,dx$$

$$= \frac{1}{4}\cdot\frac{2}{3}\left(1+4x^2\right)^{3/2}\Big]_0^1$$

$$= \frac{5\sqrt{5}-1}{6}$$

Example 2 SOLUTION

- On C_2, we choose y as the parameter.
  - So, the equations of C_2 are
    $x = 1 \quad y = 1 \quad 1 \le y \le 2$
    and

$$\int_{C_2} 2x\,ds$$

$$= \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy$$

$$= \int_1^2 2\,dy = 2$$

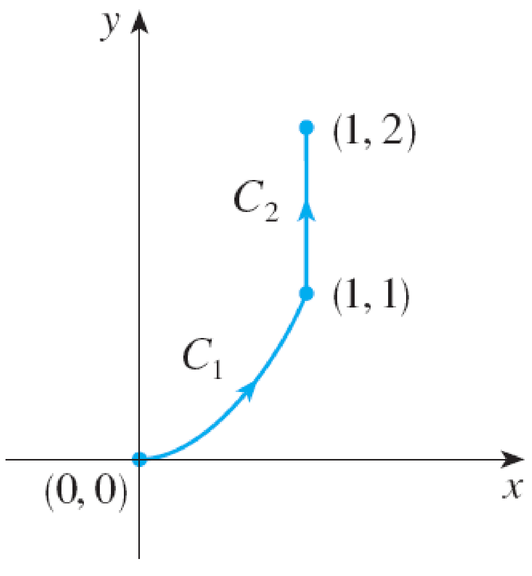

FIGURE 5
$C = C_1 \cup C_2$

Example 2 SOLUTION

- Thus,

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$$= \frac{5\sqrt{5}-1}{6} + 2$$

LINE INTEGRALS

- Any physical interpretation of a line integral

$$\int_C f(x, y)\, ds$$

depends on the physical interpretation of the function *f*.

- Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C.

Example 3

- A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geq 0$, and is thicker near its base than near the top.
  - Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Example 3 SOLUTION

- As in Example 1, we use the parametrization

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le \pi$$

and find that $ds = dt$.

Example 3 SOLUTION

- The linear density is

$$\rho(x, y) = k(1 - y)$$

where k is a constant.

- So, the mass of the wire is:

$$m = \int_C k(1 - y)\,ds = \int_0^{\pi} k(1 - \sin t)\,dt$$

$$= k\left[t + \cos t\right]_0^{\pi}$$

$$= k(\pi - 2)$$

Example 3 SOLUTION

- From Equations 4, we have:

$$\bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds = \frac{1}{k(\pi-2)}\int_C y\,k(1-y)\,ds$$

$$= \frac{1}{\pi-2}\int_0^{\pi}\left(\sin t - \sin^2 t\right)dt$$

$$= \frac{1}{\pi-2}\left[-\cos t - \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{\pi}$$

$$= \frac{4-\pi}{2(\pi-2)}$$

Example 3 SOLUTION

- By symmetry, we see that $\bar{x} = 0$.
- So, the center of mass is:

$$\left(0,\ \frac{4-\pi}{2(\pi-2)}\right) \approx (0,\ 0.38)$$

- See Figure 6.

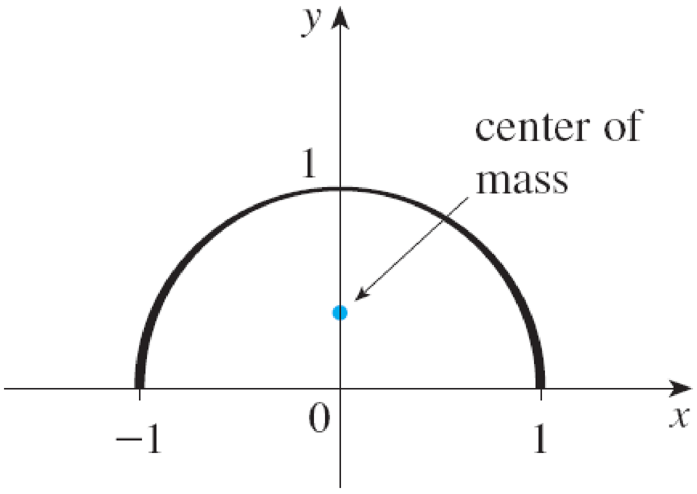

FIGURE 6

LINE INTEGRALS

- Two other line integrals are obtained by replacing Δs_i, in Definition 2, by either:
  - $\Delta x_i = x_i - x_{i-1}$
  - $\Delta y_i = y_i - y_{i-1}$

LINE INTEGRALS

❖ They are called the **line integrals of *f* along *C* with respect to *x* and *y***:

$$\int_C f(x,y)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i$$

$$\int_C f(x,y)\,dy = \lim_{\max \Delta y_j \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i$$

ARC LENGTH

- When we want to distinguish the original line integral $\int_C f(x, y)\,ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length.**

TERMS OF *t*

- The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t:

$$x = x(t)$$
$$y = y(t)$$
$$dx = x'(t)\,dt$$
$$dy = y'(t)\,dt$$

Formulas 7

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

ABBREVIATING

- It frequently happens that line integrals with respect to *x* and *y* occur together.
  - When this happens, it's customary to abbreviate by writing

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$
$$= \int_C P(x,y)\,dx + Q(x,y)\,dy$$

LINE INTEGRALS

- When we are setting up a line integral, sometimes, the most difficult thing is to think of a parametric representation for a curve whose geometric description is given.
  - In particular, we often need to parametrize a line segment.

VECTOR REPRESENTATION

- So, it's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \quad 0 \le t \le 1$$

  - See Equation 4 in Section 10.5

Example 4

- Evaluate $\int_C y^2 dx + x\,dy$ where (a) $C = C_1$ is the line segment from (–5, 3) to (0, 2) (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from (–5, 3) to (0, 2).
- See Figure 7.

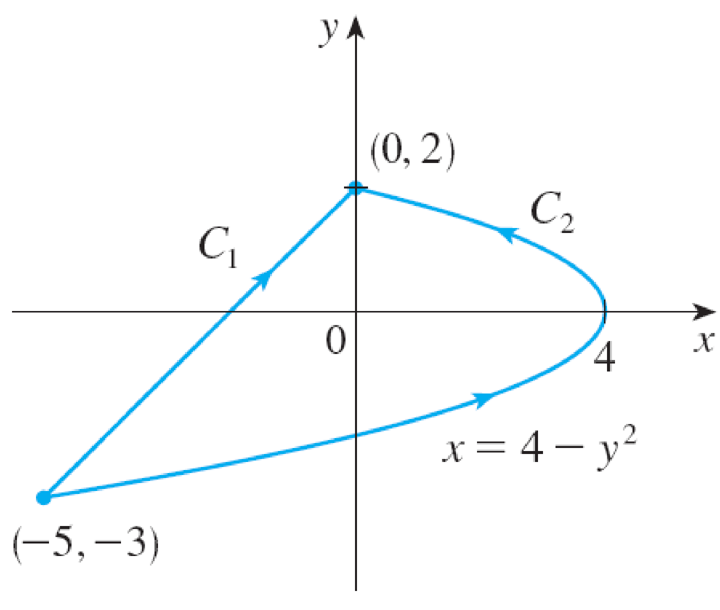

FIGURE 7

Example 4(a) SOLUTION

- A parametric representation for the line segment is:

$$x = 5t - 5 \quad y = 5t - 3 \quad 0 \le t \le 1$$

  - Use Equation 8 with $\mathbf{r}_0 = \langle -5, 3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.

Example 4(a) SOLUTION

❖Then, $dx = 5\ dt$, $dy = 5\ dt$, and Formulas 7 give:

$$\int_{C_1} y^2 dx + x\,dy = \int_0^1 (5t-3)^2 (5\,dt) + (5t-5)(5\,dt)$$

$$= 5\int_0^1 \left(25t^2 - 25t + 4\right) dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}$$

Example 4(b) SOLUTION

- The parabola is given as a function of y.
- So, let's take y as the parameter and write C_2 as:

$$x = 4 - y^2 \quad y = y \quad -3 \le y \le 2$$

Example 4(b) SOLUTION

❖ Then, $dx = -2y\ dy$ and, by Formulas 7, we have:

$$\int_{C_2} y^2 dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + \left(4 - y^2\right)dy$$

$$= \int_{-3}^{2} \left(-2y^3 - y^2 + 4\right)dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^{2} = 40\tfrac{5}{6}$$

ARC LENGTH

- Notice that we got different answers in parts a and b of Example 4 although the two curves had the same endpoints.
  - Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path.
  - However, see Section 13.3 for conditions under which it is independent of the path.

ARC LENGTH

- Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve.
  - If $-C_1$ denotes the line segment from (0, 2) to (–5, –3), you can verify, using the parametrization

$$x = -5t \qquad y = 2 - 5t \qquad 0 \le t \le 1$$

that

$$\int_{-C_1} y^2\,dx + x\,dy = \frac{5}{6}$$

CURVE ORIENTATION

- In general, a given parametrization

$$x = x(t),\ y = y(t),\ a \le t \le b$$

determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t.

CURVE ORIENTATION

- For instance, See Figure 8
  - The initial point A corresponds to the parameter value.
  - The terminal point B corresponds to $t = b$.

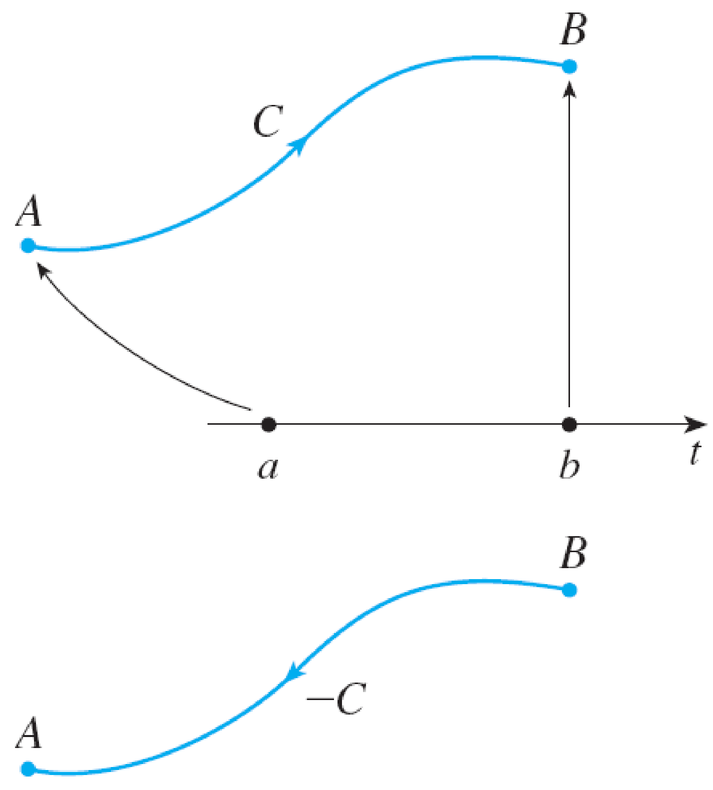

FIGURE 8

CURVE ORIENTATION

- If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), we have:

$$\int_{-C} f(x,y)\,dx = -\int_{C} f(x,y)\,dx$$

$$\int_{-C} f(x,y)\,dy = -\int_{C} f(x,y)\,dy$$

CURVE ORIENTATION

- However, if we integrate with respect to arc length, the value of the line integral does not change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\, ds = \int_{C} f(x, y)\, ds$$

  - This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

LINE INTEGRALS IN SPACE

- We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

or by a vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

LINE INTEGRALS IN SPACE

- Suppose f is a function of three variables that is continuous on some region containing C.
  - Then, we define the **line integral of *f* along *C*** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x,y,z)\,ds = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} f\left(x_i^*, y_i^*, z_i^*\right)\Delta s_i$$

LINE INTEGRALS IN SPACE

- We evaluate it using a formula similar to Formula 3:

$$\int_C f(x,y,z)\,ds$$

$$= \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

LINE INTEGRALS IN SPACE

- Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$$

LINE INTEGRALS IN SPACE

- For the special case $f(x, y, z) = 1$, we get:

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\, dt = L$$

where L is the length of the curve C.

- See Formula 3 in Section 10.8

LINE INTEGRALS IN SPACE

- Line integrals along C with respect to x, y, and z can also be defined.
  - For example,

$$\int_C f(x,y,z)\,dz = \lim_{\max \Delta z_i \to 0} \sum_{i=1}^{n} f\left(x_i^*, y_i^*, z_i^*\right)\Delta z_i$$

$$= \int_a^b f\left(x(t), y(t), z(t)\right) z'(t)\,dt$$

LINE INTEGRALS IN SPACE

- Thus, as with line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

Example 5

- Evaluate $\int_C y \sin z \, ds$

  where C is the circular helix given by the equations

  $$x = \cos t$$
  $$y = \sin t$$
  $$z = t$$

  $0 \le t \le 2\pi$

- See Figure 9.

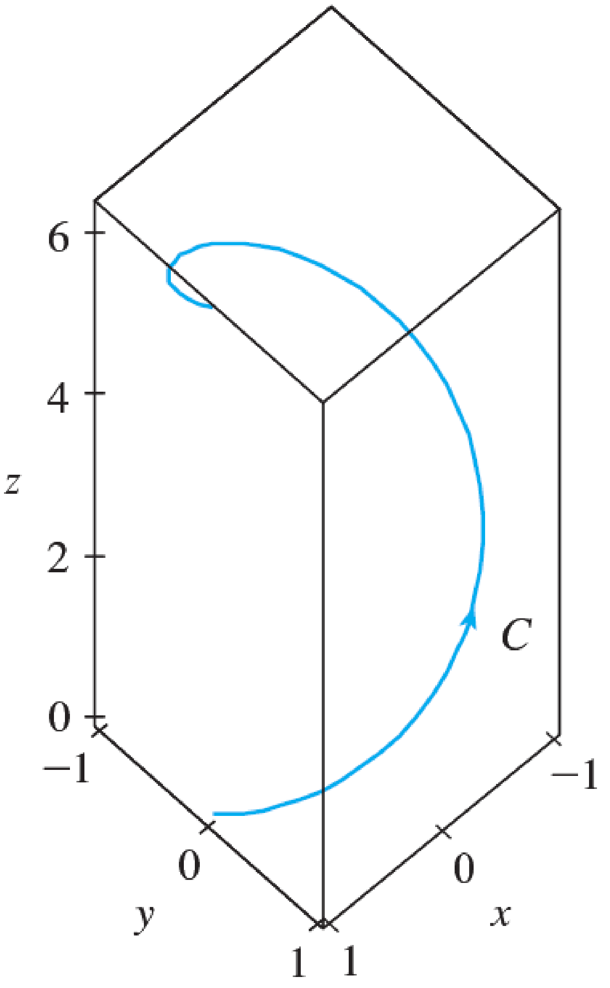

FIGURE 9

Example 5 SOLUTION

- Formula 9 gives:

$$\int_C y \sin z \, ds$$

$$= \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} \, dt$$

$$= \sqrt{2} \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t) \, dt$$

$$= \frac{\sqrt{2}}{2} \left[t - \tfrac{1}{2} \sin 2t\right]_0^{2\pi} = \sqrt{2}\pi$$

Example 6

- Evaluate

$$\int_C y\,dx + z\,dy + x\,dz$$

where C consists of the line segment C_1 from (2, 0, 0) to (3, 4, 5), followed by the vertical line segment C_2 from (3, 4, 5) to (3, 4, 0).

Example 6 SOLUTION

- The curve C is shown in Figure 10.
  - Using Equation 8, we write C_1 as:

$$r(t) = (1 - t)\langle 2, 0, 0\rangle + t\,\langle 3, 4, 5\rangle$$
$$= \langle 2 + t, 4t, 5t\rangle$$

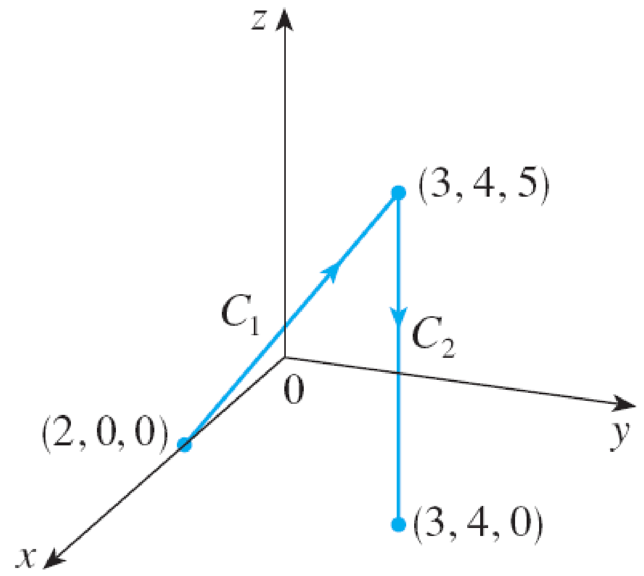

FIGURE 10

Example 6 SOLUTION

- Alternatively, in parametric form, we write C_1 as:

$$x = 2 + t$$
$$y = 4t$$
$$z = 5t$$
$$0 \le t \le 1$$

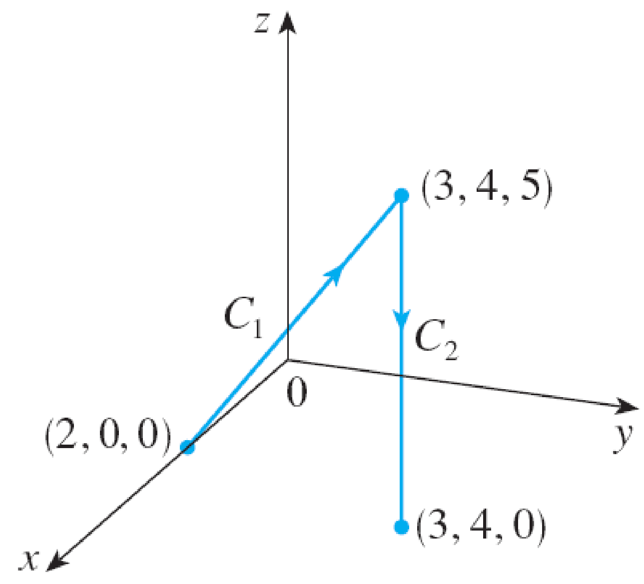

FIGURE 10

Example 6 SOLUTION

- Thus,

$$\int_{C_1} y\,dx + z\,dy + x\,dz$$

$$= \int_0^1 (4t)\,dt + (5t)\,4\,dt + (2+t)\,5\,dt$$

$$= \int_0^1 (10 + 29t)\,dt$$

$$= 10t + 29\frac{t^2}{2}\Big]_0^1 = 24.5$$

Example 6 SOLUTION

- Likewise, C_2 can be written in the form

$$r(t) = (1 - t)\langle 3, 4, 5\rangle + t\langle 3, 4, 0\rangle$$
$$= \langle 3, 4, 5 - 5t\rangle$$

or

$$x = 3 \qquad y = 4 \qquad z = 5 - 5t \qquad 0 \le t \le 1$$

Example 6 SOLUTION

- Then, $dx = 0 = dy$.

$$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 3(-5)\,dt = -15$$

- So,
  - Adding the values of these integrals, we obtain:

$$\int_{C_1} y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5$$

LINE INTEGRALS OF VECTOR FIELDS

- Recall from Section 7.5 that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is:

$$W = \int_a^b f(x)\, dx$$

LINE INTEGRALS OF VECTOR FIELDS

- In Section 10.3, we found that the work done by a constant force **F** in moving an object from a point *P* to another point in space is:

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

LINE INTEGRALS OF VECTOR FIELDS

- Now, suppose that

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

is a continuous force field on $\mathbb{R}^3$, such as:

- The gravitational field of Example 4 in Section 13.1
- The electric force field of Example 5 in Section 13.1

LINE INTEGRALS OF VECTOR FIELDS

- A force field on $\mathbb{R}^3$ could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.
  - We wish to compute the work done by this force in moving a particle along a smooth curve C.

- We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width.

LINE INTEGRALS OF VECTOR FIELDS

- Figure 1 shows the two-dimensional case.
- The second shows the three-dimensional one.

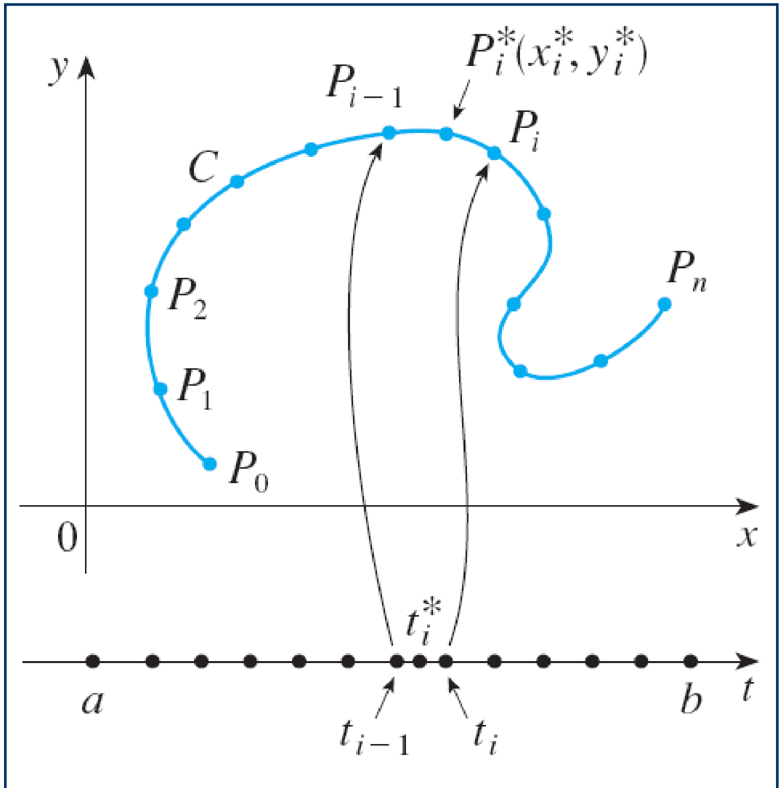

FIGURE 1

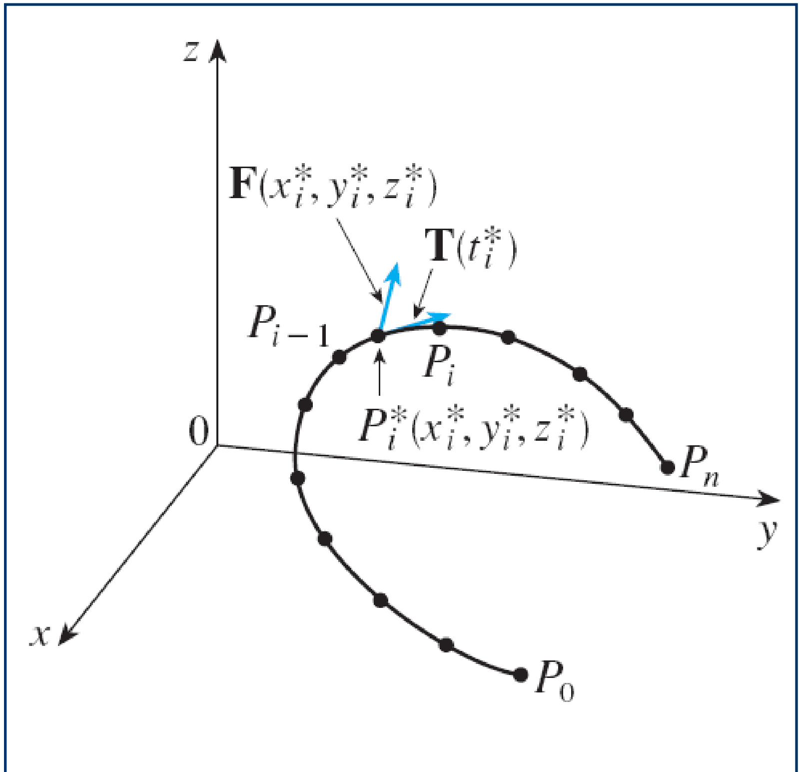

FIGURE 11

LINE INTEGRALS OF VECTOR FIELDS

❖ Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc corresponding to the parameter value t_i^*.

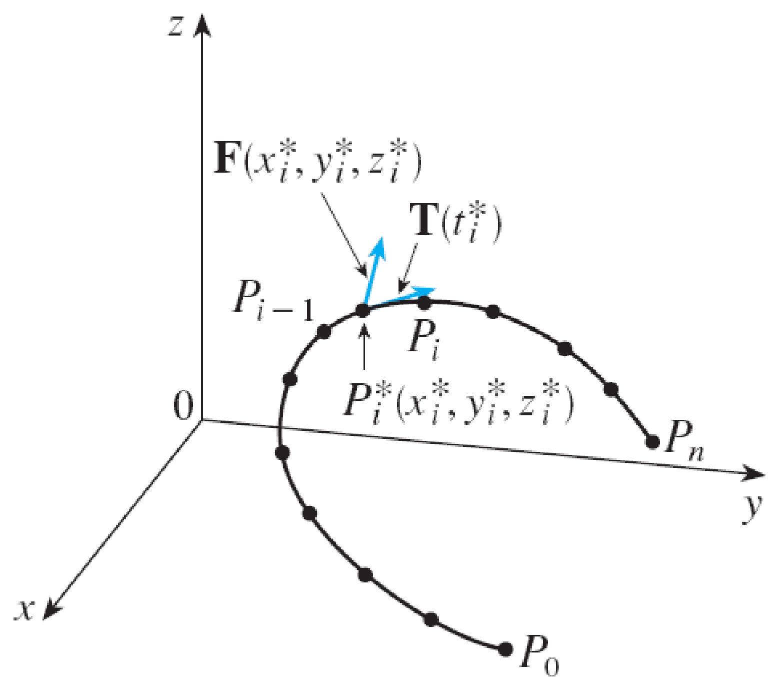

FIGURE 11

LINE INTEGRALS OF VECTOR FIELDS

- If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^*.

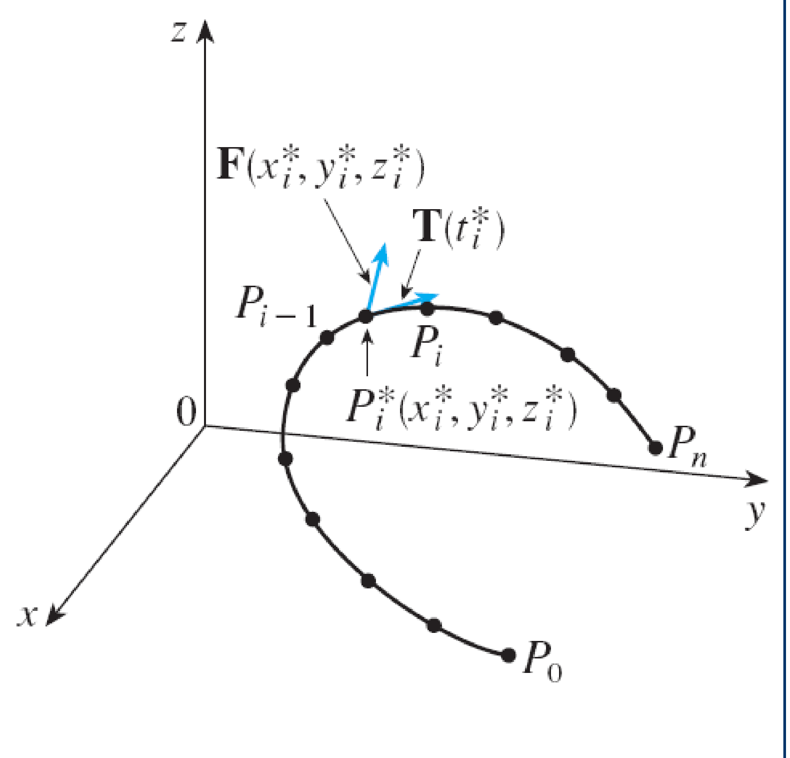

FIGURE 11

LINE INTEGRALS OF VECTOR FIELDS

- Thus, the work done by the force **F** in moving the particle P_{i-1} from to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \, \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \, \Delta s_i$$

VECTOR FIELDS

- The total work done in moving the particle along C is approximately

$$\sum_{i=1}^{n}\left[\mathbf{F}(x_i^*, y_i^*, z_i^*)\cdot\mathbf{T}(x_i^*, y_i^*, z_i^*)\right]\Delta s_i$$

  where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C.

- Intuitively, we see that these approximations ought to become better as n becomes larger.

VECTOR FIELDS

- Thus, we define the **work** W done by the force field **F** as the limit of the Riemann sums in Formula 11, namely,

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

  - This says that *work is the line integral with respect to arc length of the tangential component of the force.*

- If the curve C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

then

$$\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$$

VECTOR FIELDS

- So, using Equation 9, we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| \, dt$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

VECTOR FIELDS

- This integral is often abbreviated as

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and occurs in other areas of physics as well.

  - Thus, we make the following definition for the line integral of any continuous vector field.

Definition 13

Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then, the **line integral of F along *C*** is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

VECTOR FIELDS

- When using Definition 13, remember $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for

$$\mathbf{F}(x(t), y(t), z(t))$$

  - So, we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting

    $$x = x(t),\ y = y(t), \text{ and } z = z(t)$$

    in the expression for $\mathbf{F}(x, y, z)$.
  - Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

Example 7

❖ Find the work done by the force field

$$\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$$

in moving a particle along the quarter-circle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \le t \le \pi/2$$

Example 7 SOLUTION

- Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\, \mathbf{i} - \cos t \sin t\, \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j}$$

Example 7 SOLUTION

- Therefore, the work done is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{\pi/2} \left(-2\cos^2 t \sin t\right) dt$$

$$= 2\frac{\cos^3 t}{3}\Bigg]_0^{\pi/2} = -\frac{2}{3}$$

VECTOR FIELDS

- Figure 12 shows the force field and the curve in Example 7.
  - The work done is negative because the field impedes movement along the curve.

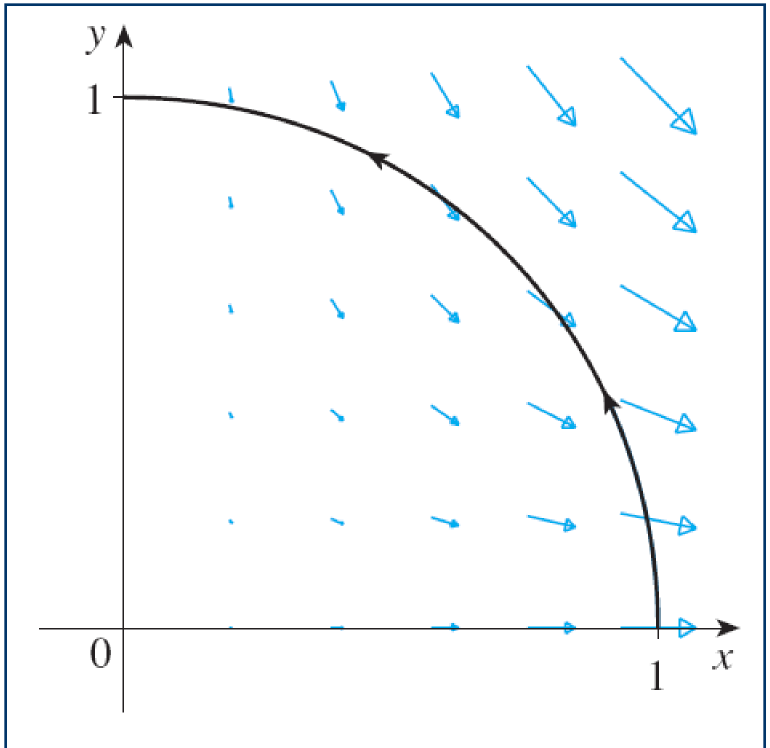

FIGURE 12

Note

- Although $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that:

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

  - This is because the unit tangent vector **T** is replaced by its negative when C is replaced by $-C$.

Example 8

- Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where:

- $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$
- C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \le t \le 1$$

Example 8 SOLUTION

- We have:

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

- Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}\left(\mathbf{r}\left(t\right)\cdot\mathbf{r}'\left(t\right)\right)dt$$

$$= \int_0^1 \left(t^3 + 5t^6\right)dt = \frac{t^4}{4} + \frac{5t^7}{7}\Big]_0^1 = \frac{27}{28}$$

VECTOR FIELDS

- Figure 13 shows the twisted cubic in Example 8 and some typical vectors acting at three points on *C*.

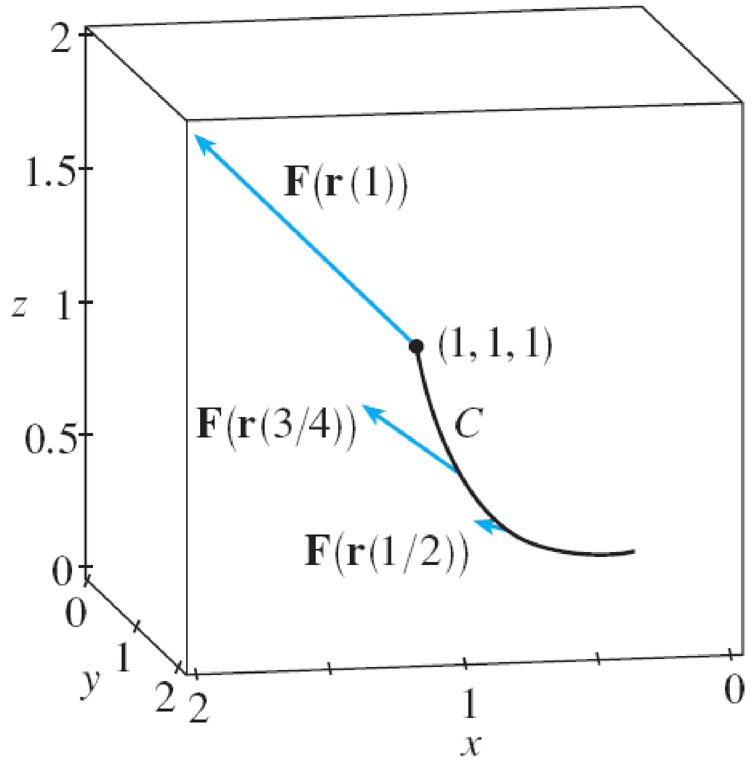

FIGURE 13

VECTOR & SCALAR FIELDS

- Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields.

VECTOR & SCALAR FIELDS

- Suppose the vector field **F** on $\mathbb{R}^3$ is given in component form by:

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

  - We use Definition 13 to compute its line integral along *C*, as follows.

VECTOR & SCALAR FIELDS

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k})\,dt$$

$$= \int_a^b \begin{bmatrix} P(x(t), y(t), z(t))\,x'(t) \\ +Q(x(t), y(t), z(t))\,y'(t) \\ +R(x(t), y(t), z(t))\,z'(t) \end{bmatrix}$$

- However, that last integral is precisely the line integral in Formula 10.
- Hence, we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$

VECTOR & SCALAR FIELDS

- For example, the integral

$$\int_C y\,dx + z\,dy + x\,dz$$

in Example 6 could be expressed as

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$