7

TECHNIQUES OF INTEGRATION

Due to the Fundamental Theorem of Calculus (FTC), we can integrate a function if we know an antiderivative, that is, an indefinite integral.

- We summarize the most important integrals we have learned so far, as follows.

FORMULAS OF INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

FORMULAS OF INTEGRALS

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 dx = \tan x + C$$

$$\int \csc^2 dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

FORMULAS OF INTEGRALS

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

In this chapter, we develop techniques for using the basic integration formulas.

- This helps obtain indefinite integrals of more complicated functions.

We learned the most important method of integration, the Substitution Rule, in Section 5.5

The other general technique, integration by parts, is presented in Section 7.1

Then, we learn methods that are special to particular classes of functions—such as trigonometric functions and rational functions.

Integration is not as straightforward as differentiation.

- There are no rules that absolutely guarantee obtaining an indefinite integral of a function.
- Therefore, we discuss a strategy for integration in Section 7.5

7.1
Integration by Parts

In this section, we will learn:

How to integrate complex functions by parts.

Every differentiation rule has a corresponding integration rule.

- For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation.

The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

INTEGRATION BY PARTS

The Product Rule states that, if *f* and *g* are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

INTEGRATION BY PARTS

In the notation for indefinite integrals, this equation becomes

$$\int \left[f(x)g'(x) + g(x)f'(x) \right] dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

INTEGRATION BY PARTS

Formula 1 is called the formula for integration by parts.

- It is perhaps easier to remember in the following notation.

Let $u = f(x)$ and $v = g(x)$.

- Then, the differentials are:

$$du = f'(x)\,dx \quad \text{and} \quad dv = g'(x)\,dx$$

Formula 2

Thus, by the Substitution Rule, the formula for integration by parts becomes:

$$\int u\,dv = uv - \int v\,du$$

Find $\int x \sin x \, dx$

- Suppose we choose $f(x) = x$ and $g'(x) = \sin x$.
- Then, $f'(x) = 1$ and $g(x) = -\cos x$.
- For g, we can choose any antiderivative of g'.

Using Formula 1, we have:

$$\int x \sin x \, dx = f(x)g(x) - \int g(x) f'(x) \, dx$$

$$= x(-\cos x) - \int (-\cos x) \, dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

- It's wise to check the answer by differentiating it.
- If we do so, we get $x \sin x$, as expected.

Let $u = x \qquad dv = \sin x\,dx$

Then, $du = dx \qquad v = -\cos x$

Using Formula 2, we have:

$$\int x\sin x\,dx = \int \overset{u}{x}\,\overbrace{\sin x\,dx}^{dv} = \overset{u}{x}\,\overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\,\overset{du}{dx}$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

NOTE

Our aim in using integration by parts is to obtain a simpler integral than the one we started with.

- Thus, in Example 1, we started with $\int x \sin x \, dx$ and expressed it in terms of the simpler integral $\int \cos x \, dx$.

NOTE

If we had instead chosen $u = \sin x$ and $dv = x\,dx$, then $du = \cos x\,dx$ and $v = x^2/2$.

So, integration by parts gives:

$$\int x \sin x \, dx = (\sin x)\frac{x^2}{2} - \frac{1}{2}\int x^2 \cos dx$$

- Although this is true, $\int x^2 \cos x \, dx$ is a more difficult integral than the one we started with.

NOTE

Hence, when choosing *u* and *dv*, we usually try to keep $u = f(x)$ to be a function that becomes simpler when differentiated.

- At least, it should not be more complicated.
- However, make sure that $dv = g'(x)\, dx$ can be readily integrated to give *v*.

Evaluate $\int \ln x \, dx$

- Here, we don't have much choice for *u* and *dv*.
- Let $u = \ln x \qquad dv = dx$
- Then, $du = \frac{1}{x} dx \qquad v = x$

Integrating by parts, we get:

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f.

Find $\int t^2 e^t dt$

- Notice that t^2 becomes simpler when differentiated.
- However, e^t is unchanged when differentiated or integrated.

So, we choose $u = t^2 \quad dv = e^t dt$

Then, $du = 2t\,dt \qquad v = e^t$

Integration by parts gives:

$$\int t^2 e^t dt = t^2 e^t - 2\int te^t dt$$

The integral that we obtained, $\int te^t dt$, is simpler than the original integral.

However, it is still not obvious.

- So, we use integration by parts a second time.

This time, we choose

$u = t$ and $dv = e^t dt$

- Then, $du = dt$, $v = e^t$.
- So, $\int te^t dt = te^t - \int e^t dt - te^t - e^t + C$

Putting this in Equation 3, we get

$$\int t^2 e^t dt = t^2 e^t - 2\int te^t dt$$

$$= t^2 e^t - 2(te^t - e^t + C)$$

$$= t^2 e^t - 2te^t - 2e^t + C_1$$

where $C_1 = -2C$

Evaluate $\int e^x \sin x \, dx$

- e^x does not become simpler when differentiated.
- Neither does sin x become simpler.

Nevertheless, we try choosing

$$u = e^x \text{ and } dv = \sin x$$

- Then, $du = e^x\,dx$ and $v = -\cos x$.

So, integration by parts gives:

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

The integral we have obtained, $\int e^x \cos x \, dx$, is no simpler than the original one.

- At least, it's no more difficult.
- Having had success in the preceding example integrating by parts twice, we do it again.

This time, we use

$u = e^x$ and $dv = \cos x\, dx$

Then, $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

At first glance, it appears as if we have accomplished nothing.

- We have arrived at $\int e^x \sin x \, dx$, which is where we started.

However, if we put the expression for $\int e^x \cos x\,dx$ from Equation 5 into Equation 4, we get:

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

- This can be regarded as an equation to be solved for the unknown integral.

Adding to both sides $\int e^x \sin x \, dx$,

we obtain:

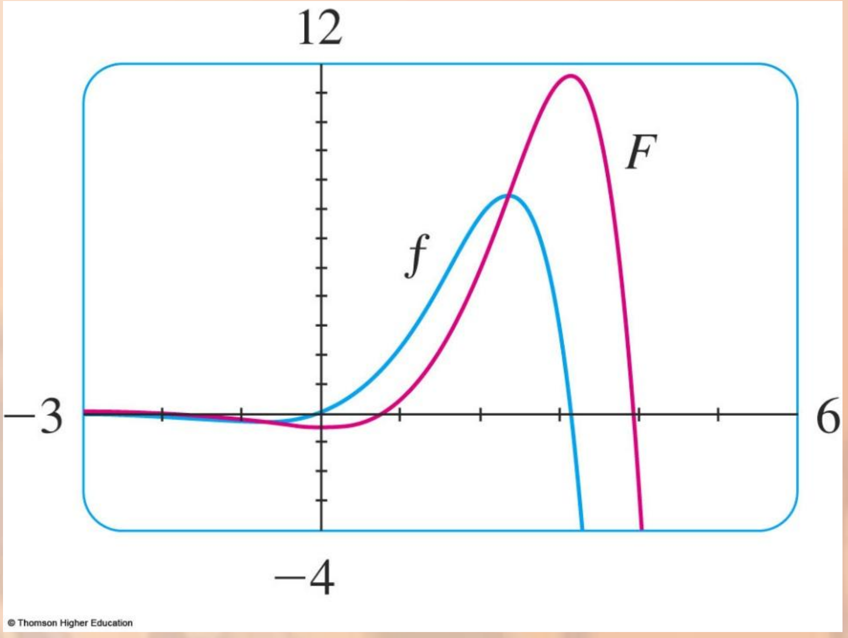
$$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

Dividing by 2 and adding the constant of integration, we get:

$$\int e^x \sin x\, dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C$$

The figure illustrates the example by showing the graphs of $f(x) = e^x \sin x$ and $F(x) = \frac{1}{2} e^x(\sin x - \cos x)$.

- As a visual check on our work, notice that $f(x) = 0$ when F has a maximum or minimum.



INTEGRATION BY PARTS

If we combine the formula for integration by parts with Part 2 of the FTC (FTC2), we can evaluate definite integrals by parts.

Formula 6

Evaluating both sides of Formula 1 between *a* and *b*, assuming *f'* and *g'* are continuous, and using the FTC, we obtain:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

Calculate $\int_0^1 \tan^{-1} x \, dx$

- Let $u = \tan^{-1} x \qquad dv = dx$
- Then, $du = \dfrac{dx}{1+x^2} \qquad v = x$

So, Formula 6 gives:

$$\int_0^1 \tan^{-1} x \, dx = x \tan^{-1} x \Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$$

$$= 1 \cdot \tan^{-1} 1 - 0 \cdot \tan^{-1} 0 - \int_0^1 \frac{x}{1+x^2}\, dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\, dx$$

To evaluate this integral, we use the substitution $t = 1 + x^2$ (since u has another meaning in this example).

- Then, $dt = 2x\,dx$.
- So, $x\,dx = \frac{1}{2}\,dt$.

When $x = 0$, $t = 1$, and when $x = 1$, $t = 2$.

Hence, $$\int_0^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \tfrac{1}{2}\ln|t|\Big]_1^2$$

$$= \tfrac{1}{2}(\ln 2 - \ln 1)$$

$$= \tfrac{1}{2}\ln 2$$

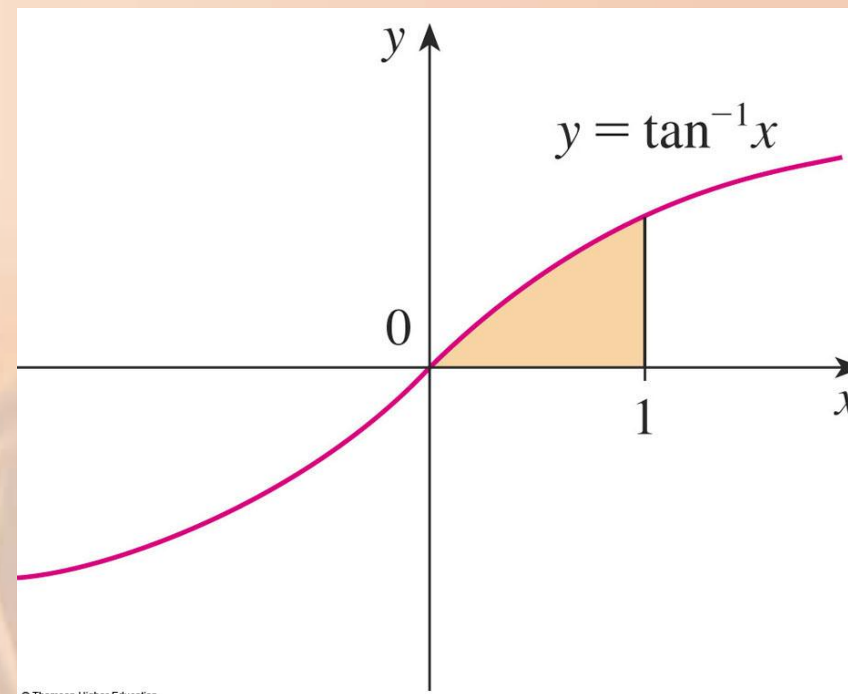
Therefore,

$$\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

INTEGRATION BY PARTS

As $\tan^{-1} x \geq$ for $x \geq 0$, the integral in the example can be interpreted as the area of the region shown here.



Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

where $n \geq 2$ is an integer.

- This is called a reduction formula because the exponent n has been reduced to $n - 1$ and $n - 2$.

Let $u = \sin^{n-1} x \qquad dv = \sin x\, dx$

Then, $du = (n-1)\sin^{n-2} x \cos x\, dx \qquad v = -\cos x$

So, integration by parts gives:

$$\int \sin^n x\, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have:

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx$$
$$-(n-1)\int \sin^n x\,dx$$

- As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side.

Thus, we have:

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n}\int \sin^{n-2} x\,dx$$

The reduction formula (7) is useful.

By using it repeatedly, we could express $\int \sin^n x \, dx$ in terms of:

- $\int \sin x \, dx$ (if n is odd)
- $\int (\sin x)^0 dx = \int dx$ (if n is even)