CHAPTER 13 VECTOR CALCULUS

STOKES' THEOREM

STOKES' VS. GREEN'S THEOREM

- Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.
  - Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve.
  - Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (a space curve).

INTRODUCTION

- Figure 1 shows an oriented surface with unit normal vector **n**.
  - The orientation of *S* induces the **positive orientation of the boundary curve *C*** shown in the figure.

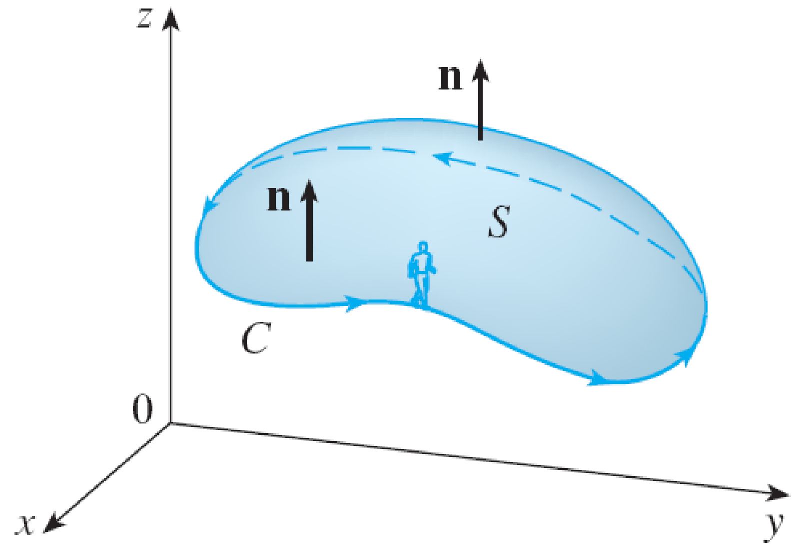

FIGURE 1

INTRODUCTION

- This means that:
  - If you walk in the positive direction around *C* with your head pointing in the direction of **n**, the surface will always be on your left.

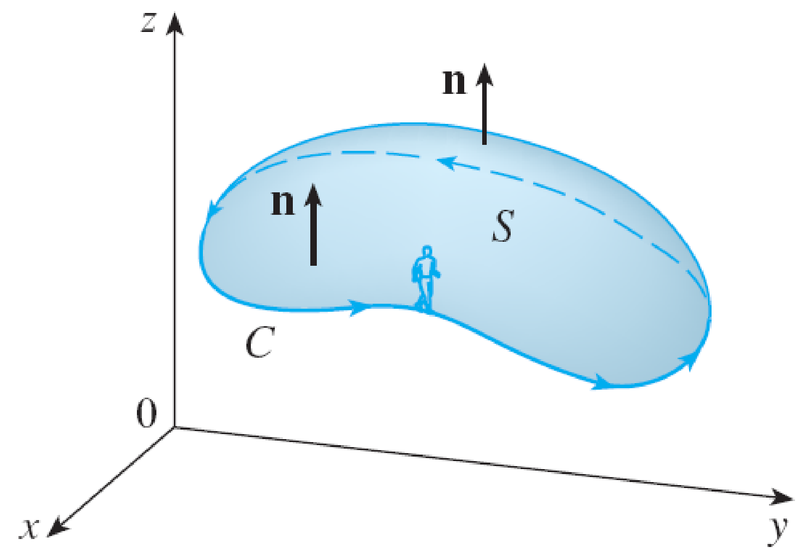

FIGURE 1

STOKES' THEOREM

- Let S be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

STOKES' THEOREM

- The theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903).
  - What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin).
  - Stokes learned of it in a letter from Thomson in 1850.

STOKES' THEOREM

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

and

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

- Thus, Stokes' Theorem says:
  - The line integral around the boundary curve of S of the tangential component of **F** is equal to the surface integral of the normal component of the curl of **F**.

STOKES' THEOREM

- The positively oriented boundary curve of the oriented surface S is often written as ∂S.
- So, the theorem can be expressed as:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

STOKES' THEOREM, GREEN'S THEOREM, & FTC

- There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus (FTC).
  - As before, there is an integral involving derivatives on the left side of Equation 1 (recall that curl **F** is a sort of derivative of **F**).
  - The right side involves the values of **F** only on the boundary of *S*.

STOKES' THEOREM, GREEN'S THEOREM, & FTC

- In fact, consider the special case where the surface *S*:
  - Is flat.
  - Lies in the *xy*-plane with upward orientation.

STOKES' THEOREM, GREEN'S THEOREM, & FTC

- Then,
  - The unit normal is **k**.
  - The surface integral becomes a double integral.
  - Stokes' Theorem becomes:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$$

STOKES' THEOREM, GREEN'S THEOREM, & FTC

- This is precisely the vector form of Green's Theorem given in Equation 12 in Section 13.5
  - Thus, we see that Green's Theorem is really a special case of Stokes' Theorem.

STOKES' THEOREM

- Stokes' Theorem is too difficult for us to prove in its full generality.
- Still, we can give a proof when:
  - S is a graph.
  - $\mathbf{F}$, S, and C are well behaved.

Example 1

- Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where:
  - $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$
  - C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

Example 1 SOLUTION

- The curve C (an ellipse) is shown in Figure 3.
  - $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly.
  - However, it's easier to use Stokes' Theorem.

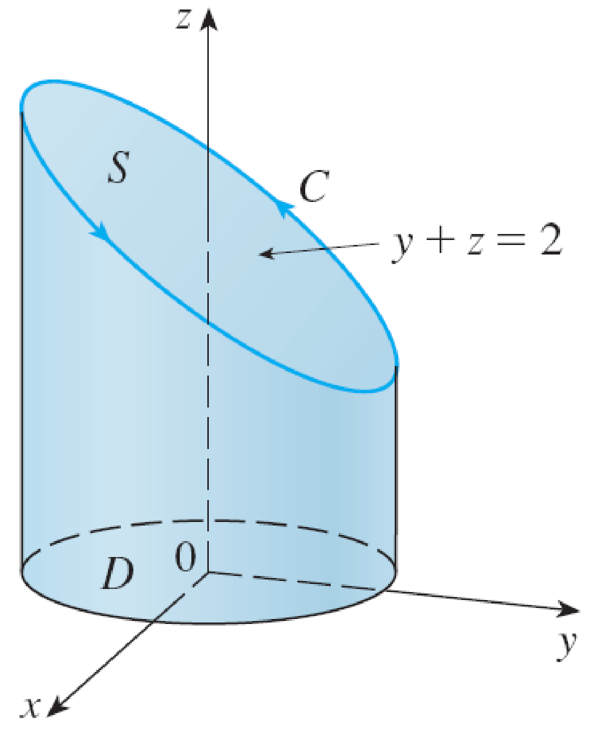

FIGURE 3

Example 1 SOLUTION

- We first compute:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)\mathbf{k}$$

Example 1 SOLUTION

- There are many surfaces with boundary *C*.
  - The most convenient choice, though, is the elliptical region *S* in the plane $y + z = 2$ that is bounded by *C*.
  - If we orient *S* upward, *C* has the induced positive orientation.

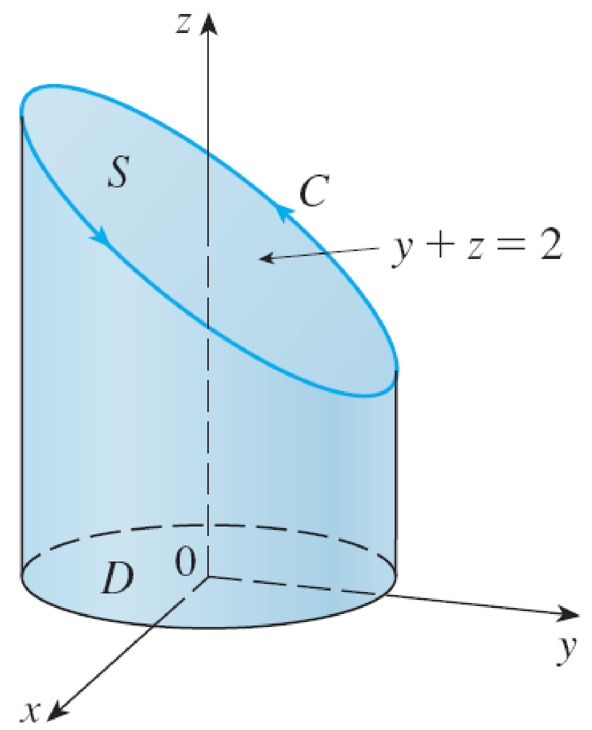

FIGURE 3

Example 1 SOLUTION

- The projection D of S on the xy-plane is the disk $x^2 + y^2 \leq 1$.
  - So, using Equation 10 in Section 13.7 with $z = g(x, y) = 2 - y$, we have the following result.

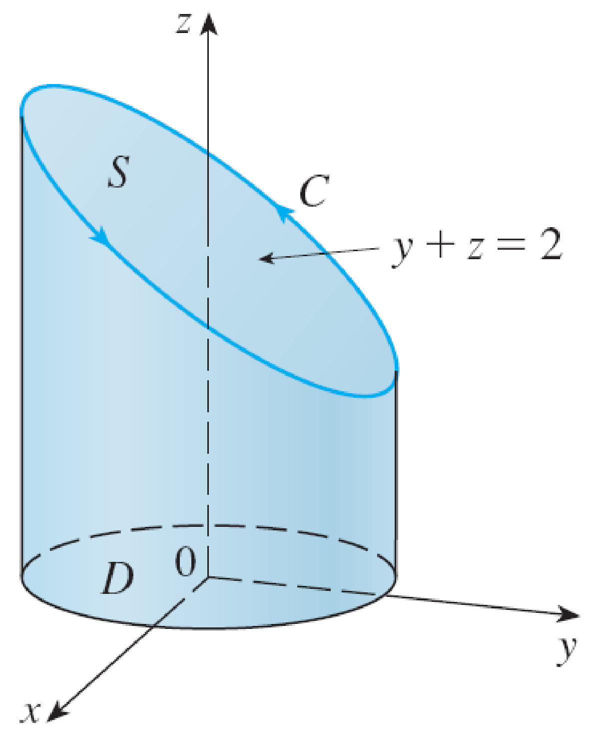

FIGURE 3

Example 1 SOLUTION

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3}\sin\theta \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin\theta \right) d\theta$$

$$= \frac{1}{2}(2\pi) + 0 = \pi$$

Example 2

- Use Stokes' Theorem to compute $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ where:
  - $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$
  - S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

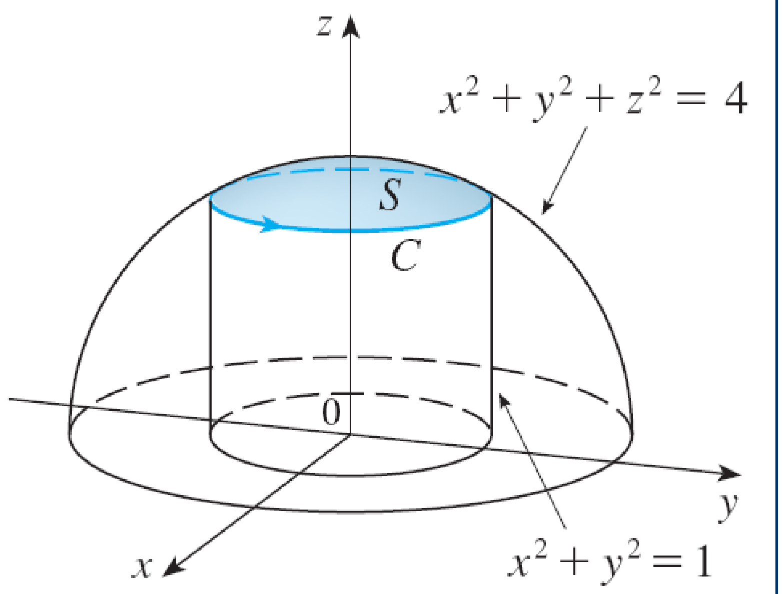

FIGURE 4